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A CLASS OF DBDF METHODS WITH THE DERIVATIVE MODIFYING TERM*1)

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Abstract

In this paper, a class of DBDF methods with the derivative modifying term is presented. The form of the methods is

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = hf_{n+k} + \alpha h^2 f'_{n+k}$$

which is of k-step and order $k+1$. The numerical stability of the new methods is much better than both Gear's methods and Enright's methods.

§1. Introduction

Since the famous thesis of G. Dahlquist——the order of any A-stable linear multistep method cannot exceed 2, and the smallest error constant is obtained for the trapezoidal rule was presented in 1963, the research of the numerical methods for stiff systems has been divided into two classes: 1) non-linear methods, such as one-leg methods and implicit Runge-Kutta methods; 2) stiff-stable linear multistep methods. The former is comparatively stable but more complex in computation while the latter is simple in the construction but weak in numerical stability. C. W. Gear introduced a class of backward differential methods (BDF) with a perfect program for automatic computation. Confined by the numerical stability, however, only the lower order methods can be used for highly oscillating systems, so it is inadequate for such problems. In this paper, we shall introduce a class of improved BDF methods which has a modifying term using the second derivative. Being assured of the zero-stability, the new methods have excellent absolute stability as compared with Gear's methods and Enright's methods which, as our new methods, contain the second derivative.

§2. The Construction of DBDF Methods

Following the notation in [5], we let D and E be the differential and displacement operator respectively, that is

$$Dy(x) = y'(x), \quad Ey(x) = y(x+h) \tag{1}$$

then the backward difference operator ∇ satisfies

$$\nabla y(x) = y(x) - y(x-h) = (I - E^{-1})y(x)$$

so that

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$$\nabla = I - E^{-1}. \tag{2}$$

By the operator formula (see [5]),

$$hD = \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \cdots = \sum_{j=1}^{\infty}\frac{1}{j}\nabla^j \tag{3}$$

we have

$$h^2D^2 = \sum_{j=2}^{\infty}\sum_{i=1}^{j-1}\frac{1}{i(j-i)}\nabla^j. \tag{4}$$

Using (3) and (4) we obtain immediately

$$hD + \alpha h^2D^2 = \nabla + \sum_{j=2}^{\infty}\left(\frac{1}{j} + \sum_{i=1}^{j-1}\frac{\alpha}{i(j-i)}\right)\nabla^j, \tag{5}$$

where α is a real parameter.

Truncating the first k terms of the right hand side of (5) and together with (2), we have

$$hDE^k + \alpha h^2D^2E^k = E^k\left\{(I - E^{-1}) + \sum_{j=2}^{k}\left(\frac{1}{j} + \sum_{i=1}^{j-1}\frac{\alpha}{i(j-i)}\right)(I - E^{-1})^j\right\} = \rho_k(E),$$

and thus we have constructed a class of modified BDF methods with derivative modifying term, i.e. DBDF methods:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = hf_{n+k} + \alpha h^2 f'_{n+k}, \tag{6}$$

with its first characteristic polynomial

$$\rho_k(\zeta) = \zeta^k\left\{(1 - \zeta^{-1}) + \sum_{j=2}^{k}\left[\frac{1}{j} + \sum_{i=1}^{j-1}\frac{\alpha}{i(j-i)}\right](1 - \zeta^{-1})^j\right\}. \tag{7}$$

It is easy to calculate the coefficients α_j in (6):

$$\alpha_k = \sum_{j=1}^{k}\frac{1}{j} + \sum_{j=2}^{k}\sum_{i=1}^{j-1}\frac{\alpha}{i(j-i)},$$

$$\alpha_{k-p} = (-1)^p\sum_{j=p}^{k}\binom{j}{p}\left[\frac{1}{j} + \sum_{i=1}^{j-1}\frac{\alpha}{i(j-i)}\right], \quad p = 1, 2, \ldots, k.$$

Let $\mathscr{L}[E; h]$ denote the difference operator of DBDF methods (6), that is

$$\begin{aligned}\mathscr{L}[E; h] &\equiv \rho_k(E) - hDE^k - \alpha h^2D^2E^k \\ &= E^k\left\{hD + \alpha h^2D^2 - \sum_{j=k+1}^{\infty}\left[\frac{1}{j} + \sum_{i=1}^{j-1}\frac{\alpha}{i(j-i)}\right]\nabla^j\right\} - hDE^k - \alpha h^2D^2E^k \\ &= -E^k\sum_{j=k+1}^{\infty}\left\{\frac{1}{j} + \sum_{i=1}^{j-1}\frac{\alpha}{i(j-i)}\right\}\nabla^j \\ &\sim -\left[\frac{1}{k+1} + \sum_{i=1}^{k}\frac{\alpha}{i(k+1-i)}\right](hD)^{k+1}, \quad h \to 0.\end{aligned}$$

Hence the error constant

$$C_{k+1} = -\frac{1}{k+1} - \sum_{i=1}^{k}\frac{\alpha}{i(k+1-i)}. \tag{8}$$

Theorem 1. *DBDF methods (6) is of order $k+1$ if and only if* $\alpha = -\left(2\sum_{j=1}^{k}\frac{1}{j}\right)^{-1}$.

Proof. By (8), DBDF methods (6) is of order $k+1$ if and only if $C_{k+1} = 0$, i.e.

$$\alpha = -\left(\sum_{i=1}^{k}\frac{k+1}{i(k+1-i)}\right)^{-1}.$$