A NEW QUADRILATERAL ELEMENT APPROXIMATION TO THE STATIONARY STOKES PROBLEM*1)

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Abstract

The nearly linear triangular element approximation to the stationary Stokes problem has already been proposed[1,7]. This paper discusses new rectangular and quadrilateral element approximations to the same problem, and the first order error is proved for both elements.

§ 1. Introduction

In [7] and [1], the nearly linear triangular element approximation to the stationary Stokes problem has been proposed. The quadrilateral finite elements are attractive for discretization of a domain of arbitrary shapes, and some quadrilateral element approximations to the Stokes problem have been studied (c.f. [2], [5]). In the present paper, some new quadrilateral elements approximations for both rectangles and general quadrilaterals are discussed, and the first order error is obtained for the Stokes problem, in the same way as in [7].

Let us consider approximation to the Stokes problem,

$$
\text{(ST)} \quad \begin{cases} \text{find } (\boldsymbol{u}, p) \in (H_0^1(\Omega))^2 \times L_0^2(\Omega), \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \ \forall \boldsymbol{v} \in (H_0^1(\Omega))^2, & (1.1) \\ \qquad b(\boldsymbol{u}, q) = 0, \qquad \forall q \in L_0^2(\Omega), & (1.2) \end{cases}
$$

where Ω is a bounded convex polygon in plane with a boundary $\partial\Omega$, and

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu(\text{grad } \boldsymbol{u}, \text{grad } \boldsymbol{v}), \tag{1.3}$$

$$b(\boldsymbol{v}, q) = -(q, \text{div } \boldsymbol{v}), \tag{1.4}$$

$(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$, and in what follows, $H_0^1(\Omega)$, $H^m(\Omega)$ denote usual Sobolev spaces with the norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$, and

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx \, dy, \tag{1.5}$$

$$L_0^2(\Omega) = \{q \in L^2(\Omega): \int_\Omega q dx \, dy = 0\}. \tag{1.6}$$

In equations (1.1) and (1.2), $\boldsymbol{u} = (u_1, u_2)^T$ denotes the velocity of fluid, p denotes the pressure and $\nu = \text{const.} > 0$ denotes the viscosity.

Let $X_h \subset (H_0^1(\Omega))^2$ and $M_h \subset L_0^2(\Omega)$ be two finite element spaces. Then the discrete analogy of the problem (ST) is the following

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$$(\mathrm{ST}_h)\quad\begin{cases}\text{find}\,(\boldsymbol{u}_h, p_h)\in X_h\times M_h, \text{ such that}\\ a(\boldsymbol{u}_h, \boldsymbol{v}_h)-(p_h, \operatorname{div}\boldsymbol{v}_h)=\langle \boldsymbol{f}, \boldsymbol{v}_h\rangle, \quad \forall \boldsymbol{v}_h\in X_h, & (1.7)\\ \qquad (q_h, \operatorname{div}\boldsymbol{u}_h)=0, \qquad \forall q_h\in M_h. & (1.8)\end{cases}$$

It is known ([3], [6]) that if the discrete inf-sup condition holds, i.e., there exists $\beta^*=\text{const.}>0$, such that

$$\sup_{0\neq v_h\in X_h}\frac{(q_h, \operatorname{div}\boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}}\geqslant\beta^*\|q_h\|_{0,\Omega}, \quad \forall q_h\in M_h, \tag{1.9}$$

then the unique solution $(\boldsymbol{u}_h, p_h)\in X_h\times M_h$ of the problem (ST_h) satisfies the following error estimate

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\|p-p_h\|_{0,\Omega}\leqslant c\{\inf_{\boldsymbol{v}_h\in X_h}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,\Omega}+\inf_{q_h\in M_h}\|p-q_h\|_{0,\Omega}\}. \tag{1.10}$$

And it is also known[6] that if for each $q_h\in M_h$, there exists a function $\boldsymbol{v}_h\in X_h$, such that

$$(\operatorname{div}\boldsymbol{v}_h-q_h, s_h)=0, \quad \forall s_h\in M_h, \tag{1.11}$$

and

$$|\boldsymbol{v}_h|_{1,\Omega}\leqslant c\|q_h\|_{0,\Omega}, \tag{1.12}$$

then the inf-sup condition (1.9) holds. Here and later c and c_i denote generic constants independent of h.

Furthermore, condition (1.9) can be verified by constructing an operator r_h: $(H_0^1(\Omega))^2\to X_h$ such that[6]

$$(\operatorname{div}\boldsymbol{v}-\operatorname{div}r_h\boldsymbol{v}, s_h)=0, \quad \forall s_h\in M_h, \ \boldsymbol{v}\in(H_0^1(\Omega))^2, \tag{1.13}$$

$$|r_h\boldsymbol{v}|_{1,\Omega}\leqslant c|\boldsymbol{v}|_{1,\Omega}, \quad \forall \boldsymbol{v}\in(H_0^1(\Omega))^2. \tag{1.14}$$

In the case of $M_h\subset L_0^2(\Omega)\cap H^1(\Omega)$, $X_h\subset(H_0^1(\Omega))^2$, condition (1.13) becomes

$$\begin{aligned}0&=\int_\Omega s_h\cdot\operatorname{div}(\boldsymbol{v}-r_h\boldsymbol{v})dx\,dy=\sum_{K\in\mathcal{T}_h}\int_K\operatorname{div}(\boldsymbol{v}-r_k\boldsymbol{v})\cdot s_h dxdy\\ &=\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\boldsymbol{v}-r_K\boldsymbol{v})^T\cdot\boldsymbol{\nu}_K\cdot s_h d\sigma-\sum_{K\in\mathcal{T}_h}\int_K(\boldsymbol{v}-r_K\boldsymbol{v})^T\cdot\operatorname{grad}s_h dxdy,\end{aligned} \tag{1.15}$$

where $r_K\boldsymbol{v}=r_h\boldsymbol{v}|_K$, and $\boldsymbol{\nu}_K$ denotes the unit outward normol vector on the boundary ∂K of K. Since $r_h\boldsymbol{v}\in X_h\subset(H_0^1(\Omega))$, $s_h\in M_h\subset H^1(\Omega)$, then $r_h\boldsymbol{v}\in(C(\bar{\Omega}))^2$, $s_h\in C^0(\bar{\Omega})$. And since $\boldsymbol{v}\in(H_0^1(\Omega))^2$, it can be easily seen that $\boldsymbol{v}$ is continuous across the element boundary ∂K of K. Summarizing the above argument, we can verify that the first term on the right-hand side of (1.15) vanishes. Thus, if $M_h\subset L_0^2(\Omega)\cap H^1(\Omega)$, $X_h\subset(H_0^1(\Omega))^2$, then condition (1.13) is equivalent to the following condition

$$\sum_{K\in\mathcal{T}_h}\int_K(\boldsymbol{v}-r_K\boldsymbol{v})^T\cdot\operatorname{grad}s_h\,dx\,dy=0, \quad \forall s_h\in M_h, \ \boldsymbol{v}\in(H_0^1(\Omega))^2. \tag{1.16}$$

§ 2. Basic Notations

Let $\mathcal{T}_h$ (with a parameter $h>0$) be a subdivision of a convex polygon Ω in plane, and for each convex quadrilateral element $K\in\mathcal{T}_h$, let $h_K=\operatorname{diam}(K)$, h'_K be the smallest length of the edges of K, and θ_i^K be the angles associated with the vertices $P_i(1\leqslant i\leqslant 4)$ of K (Fig. 1). We assume that $\mathcal{T}_h$ satisfies the following regularity condition ([4], p. 247): There exist positive constants σ and γ, such that $\forall K\in\mathcal{T}_h$,