HIERARCHICAL ELEMENTS, LOCAL MAPPINGS AND THE *H-P* VERSION OF THE FINITE ELEMENT METHOD (II)*

GUI WEN-ZHUANG (桂文庄)
(Computing Center, Academia Sinica, Beijing, China)

Abstract

This is the second half of the article. The rate of convergence for the h-p version with geometric meshes is discussed.

§ 3. C^0-compatible Local Mappings and Geometric Meshes

In this section we discuss two C^0-compatible local mappings and the geometric meshes which utilize these mappings.

3.1. The Bilinear Mapping

The simplest mapping which maps the standard element $D=[-1, 1]\times[-1, 1]$ to an arbitrary quadrilateral element E is the bilinear mapping

$$\begin{cases} x=a_1\xi+b_1\eta+c_1\xi\eta+d_1, \\ y=a_2\xi+b_2\eta+c_2\xi\eta+d_2. \end{cases} \tag{3.1.1}$$

Suppose the vertices of the quadrilaterals E and D are numbered counter-clockwise as shown in Fig. 3.1.1:

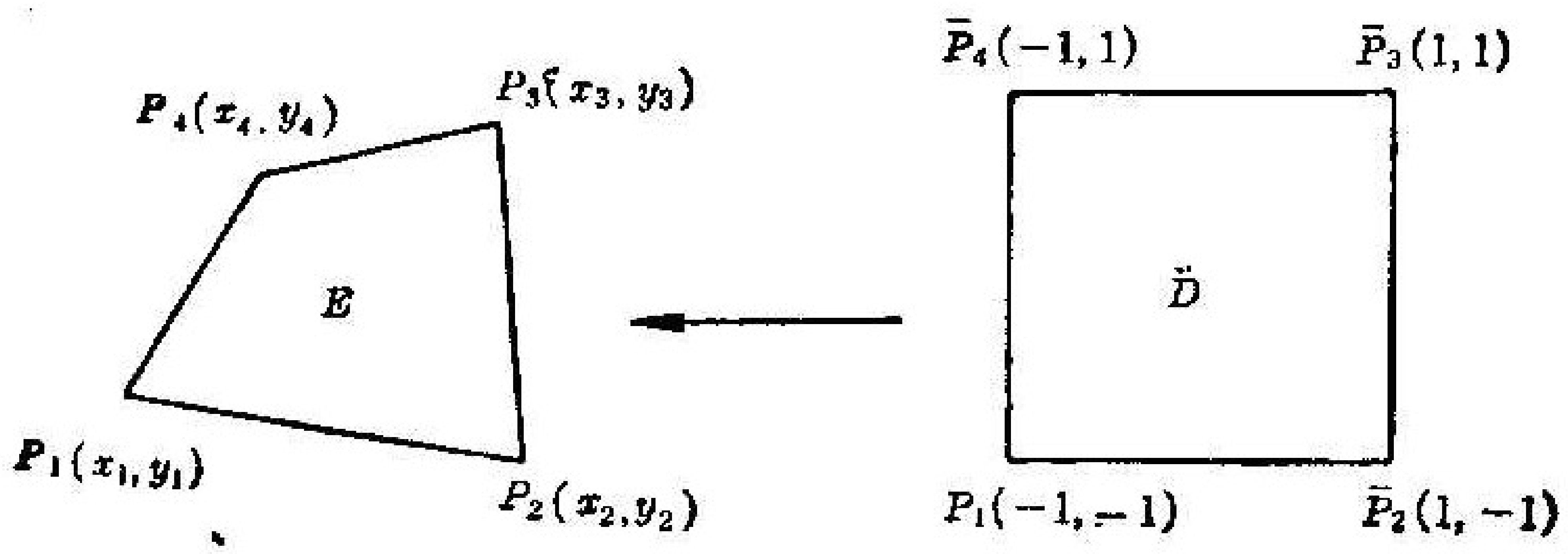

Fig. 3.1.1

Then we have

$$a_1=\frac{1}{4}(-x_1+x_2+x_3-x_4), \quad a_2=\frac{1}{4}(-y_1+y_2+y_3-y_4),$$

$$b_1=\frac{1}{4}(-x_1-x_2+x_3+x_4), \quad b_2=\frac{1}{4}(-y_1-y_2+y_3+y_4),$$

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$$c_1=\frac{1}{4}(x_1-x_2+x_3-x_4), \qquad c_2=\frac{1}{4}(y_1-y_2+y_3-y_4),$$

$$d_1=\frac{1}{4}(x_1+x_2+x_3+x_4), \qquad d_2=\frac{1}{4}(y_1+y_2+y_3+y_4). \tag{3.1.2}$$

The Jacobian of this mapping is

$$\frac{\partial(x, y)}{\partial(\xi, \eta)}=A\xi+B\eta+C, \tag{3.1.3}$$

where

$$A=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}=\frac{1}{4}(S_{123}-S_{124}),$$

$$B=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}=\frac{1}{4}(S_{134}-S_{124}),$$

$$C=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=\frac{1}{4}S_{1234}$$

in which S_{ijk} is the area of the triangle $P_iP_jP_k$, and S_{1234} is the area of the quadrilateral $E=P_1P_2P_3P_4$.

It is easy to show that the Jacobian evaluated at each vertex $\bar{P}_i$ of D equals half the area of the triangle which is determined by the corresponding vertex P_i of E with its two adjacent vertices. Thus we have (see (2.3.7) in [1])

$$C^2=\max\left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right|=\frac{1}{2}\max\{S_{123}, S_{234}, S_{341}, S_{124}\}, \tag{3.1.4}$$

$$\delta^2=\min\left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right|=\frac{1}{2}\min(S_{123}, S_{234}, S_{341}, S_{124}\}. \tag{3.1.5}$$

The only bilinear mapping with a constant Jacobian is the one which maps D to a parallelogram. In this case the mapping is

$$\begin{cases} x=a_1\xi+b_1\eta+d_1, \\ y=a_2\xi+b_2\eta+d_2 \end{cases} \tag{3.1.6}$$

with

$$\frac{\partial(x, y)}{\partial(\xi, \eta)}=C=\frac{1}{4}S_{1234}.$$

The simplest case that the mapping maps D to a rectangle was discussed in Theorem 2.3.2 (see [1]).

It is easy to show that the bilinear mapping on arbitrary quadrilateral meshes given by (3.1.1) is C^0-compatible.

3.2. The Polar Mapping

Using polar coordinates we can transform the polar net in (x, y)-plane to the rectangular net in (r, θ)-plane, then a linear mapping transforms the elements to the standard square. It is clear that this mapping is C^0-compatible.

The local mapping is the composition of the following: