A SYMPLECTIC DIFFERENCE SCHEME FOR THE INFINITE DIMENSIONAL HAMILTON SYSTEM*1)

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Abstract

Symplectic geometry plays a very important role in the research and development of Hamilton mechanics, which has been attracting increasing interest[1],[4]. Consequently, the study of the numerical methods with symplectic nature becomes a necessity.

Feng Kang introduced in [5] the concept of symplectic scheme of the Hamilton equation, and used the generating function methods to construct the symplectic scheme with arbitrarily precise order in the finite dimensional case, which can be applied to the ordinary differential equation, such as the two body problem[1]. He also widened the traditional concept of generating function.

The authors in this paper use the method in the infinite dimensional case following [6], that is, using generating function methods to construct the difference scheme of arbitrary order of accuracy for partial differential equations which can be written as Hamilton system in the Banach space.

First, the Hamilton equation of infinite dimensions is briefly reviewed. Then, we introduce symplectic manifold and symplectic structure in the Banach space. Thirdly, Hamilton vector field and its flow are discussed. Fourthly, we put forward the generating functional and symplectic difference scheme. Fifthly, the application of this result in various Banach spaces, such as Toda lattice equation, wave equation, compressible flow equation and electromagnetic flow equation, is described.

§ 1. An Infinite Dimensional Hamilton Equation

Suppose B is a reflexive Banach space and B^* its dual space. E^n is an Euclidean space and n its dimension. The generalized coordinate in the Banach space is function $q(r, t)$: $E^n\times R\rightarrow R$, $\forall t\in R$. We have $q(r, t)\in B$. B corresponds to the configuration space. We introduce $p(r, t)$, the generalized moment, where $r\in E^n$, $t\in R$. For $\forall t\in R$, $P(r, t)\in B^*$. $B\times B^*$ constitutes a phase space.

Let H be an energy function in Hamilton mechanics. We have the Hamilton equation in $B\times B^*$[2]

$$\begin{aligned}\frac{dp}{dt}&=-\frac{\delta H}{\delta q}(p, q, t),\\ \frac{dq}{dt}&=\frac{\delta H}{\delta p}(p, q, t).\end{aligned}\tag{1}$$

Let T be a mapping: $B\times B^*\rightarrow B\times B^*$. We call T the canonical transformation, if in $D\subset B\times B^*$, where D is an open set, we have

$$\int p\,dq-\int P\,dQ=dS(p, q, t),\tag{2}$$

where $\begin{pmatrix}P\\Q\end{pmatrix}=T\begin{pmatrix}p\\q\end{pmatrix}$ and S is a function in $B\times B^*$, which has Frecht derivative in D.

* Received December 10, 1986.
1) The Projectis Supported by National Natural Science Foundation of China.

Let

$$K(P, Q, t) = H + \frac{\partial S}{\partial t} + \int P \frac{dQ}{dt} d_\mu \Big|_{(p,q)\to(P,Q)}. \tag{3}$$

Considering (2), we have

$$\int p\,dq - H\,dt = \int P\,dQ - K\,dt + dS. \tag{4}$$

In the new variable, the Hamilton equation has the form:

$$\dot{P} = -\frac{\delta K}{\delta Q}, \qquad \dot{Q} = \frac{\delta K}{\delta P}. \tag{5}$$

Suppose the moment P can be represented in q and Q_2. The functional $S_1 = S(q, P(q, Q, t), t)$ is called generating functional, when $\frac{\delta^2 S_1}{\delta q \delta Q}$ is nonsingular.

From (4), we have

$$p = \frac{\delta}{\delta q} S_1(q, Q, t), \qquad P = \frac{-\delta}{\delta Q} S_1(q, Q, t). \tag{6}$$

Example. Let $S_1(q, Q) = \int_D q \cdot Q d_\mu$. Then

$$p = \frac{+\delta S_1}{\delta q} = Q,$$

$$P = \frac{-\delta S_1}{\delta Q} = -q,$$

$$K(P, Q, t) = H(Q, -P, t).$$

There are other kinds of generating functional: $S_2(p, Q, t)$, $S_3(q, P, t)$ and $S_4(p, P, t)$. For convenience, we give a list below:

G-F	Nonsingular condition	New variable
$S_1(q, Q, t)$	$\frac{\delta^2 S_1}{\delta q \delta Q} \neq 0$	$p = \frac{\delta S_1}{\delta q}(q, Q, t)$ $P = -\frac{\delta S_1}{\delta Q}(q, Q, t)$
$S_2(p, Q, t)$	$\frac{\delta^2 S_2}{\delta p \delta Q} \neq 0$	$q = -\frac{\delta S_2}{\delta p}(p, Q, t)$ $P = -\frac{\delta S_2}{\delta Q}(p, Q, t)$
$S_3(q, P, t)$	$\frac{\delta^2 S_3}{\delta q \delta P} \neq 0$	$p = \frac{\delta S_3}{\delta q}(q, Q, t)$ $Q = \frac{\delta S_3}{\delta P}(q, P, t)$
$S_4(p, P, t)$	$\frac{\delta^2 S_4}{\delta p \delta P} \neq 0$	$q = \frac{\delta S_4}{\delta p}(p, P, t)$ $Q = -\frac{\delta S_4}{\delta P}(p, P, t)$