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A NOTE ON SIMPLE NON-ZERO SINGULAR VALUES*[1)]

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Abstract

The technique described in [4] is used to investigate the analyticity and to obtain second order perturbation expansions of simple non-zero singular values of a matrix analytically dependent on several parameters.

The object of this note is to use the technique described in [4] to investigate the analyticity and to obtain second order perturbation expansions of simple non-zero singular values of a matrix analytically dependent on several parameters. The results may be useful for investigating the performance and robustness of multivariable feedback systems as well as design techniques (see [3] and the references contained therein).

Notation. The symbol $\mathbb{C}^{m\times n}$ denotes the set of complex $m\times n$ matrices and $\mathbb{R}^{m\times n}$ the set of real $m\times n$ matrices, $\mathbb{C}^n=\mathbb{C}^{n\times 1}$ and $\mathbb{R}=\mathbb{R}^1$. The superscript H is for conjugate transpose, and T for transpose. $\|x\|$ denotes the usual Euclidean vector norm of x and $\|A\|$ denotes the spectral norm of a matrix A.

§1. Singular Values of a Complex Matrix

Let $p=(p_1, \cdots, p_N)^T$ and $A(p)\in\mathbb{C}^{m\times n}$. We may assume without loss of generality that the parameters $p_1, \cdots, p_N$ are real and $m\geqslant n$ throughout this note.

Let $A=A(p^*)$ for some point $p^*\in\mathbb{R}^N$. Suppose that σ is a singular value of A. Then there exist two unit vectors, $v\in\mathbb{C}^n$ and $u\in\mathbb{C}^m$, such that

$$Av=\sigma u, \quad A^Hu=\sigma v.$$

Such v, u will be called unit right and unit left singular vectors of A corresponding to the singular value σ.

First, applying the Implicit Function Theorem we prove the following theorem.

Theorem 1.1. *Let $p\in\mathbb{R}^n$ and $A(p)\in\mathbb{C}^{m\times n}$. Suppose that* $\mathrm{Re}[A(p)]$ *and* $\mathrm{Im}[A(p)]$ *are real analytic matrix-valued functions of p in some neighbourhood $\mathscr{B}(0)$ of the origin. If σ_1 is a simple non-zero singular value of $A(0)$, $v_1\in\mathbb{C}^n$ and $u_1\in\mathbb{C}^m$ are associated unit right and unit left singular vectors, respectively, then*

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1) *there exists a simple singular value* $\sigma_1(p)$ *of* $A(p)$ *which is a real analytic function of* p *in some neighbourhood* $\mathscr{B}_0$ *of the origin, and* $\sigma_1(0)=\sigma_1$;

2) *the unit right singular vector* $v_1(p)$ *and the unit left singular vector* $u_1(p)$ *of* $A(p)$ *corresponding to* $\sigma_1(p)$ *may be so defined that* $\mathrm{Re}[v_1(p)]$, $\mathrm{Im}[v_1(p)]$, $\mathrm{Re}[u_1(p)]$ *and* $\mathrm{Im}[u_1(p)]$ *are real analytic functions of* p *in* $\mathscr{B}_0$, $v_1(0)=v_1$ *and* $u_1(0)=u_1$.

Proof. By hypotheses there exist two unitary matrices

$$U=(u_1, U_2)\in\mathbb{C}^{m\times m},\quad V=(v_1, V_2)\in\mathbb{C}^{n\times n} \tag{1.1}$$

such that

$$U^H A(0) V=\Sigma=\begin{pmatrix}\sigma_1 & 0\\ 0 & \Sigma_2\end{pmatrix},\quad \Sigma_2=\begin{pmatrix}\sigma_2 & & \\ & \ddots & \\ & & \sigma_n\\ & 0 & \end{pmatrix}\in\mathbb{R}^{(m-1)\times(n-1)}, \tag{1.2}$$

where $\sigma_1, \cdots, \sigma_n\geqslant 0$ and $\sigma_j\neq\sigma_1>0$ for $j=2, \cdots, n$.

We set

$$\tilde{A}(p)=V^H A(p)^H A(p) V=\begin{pmatrix}\tilde{a}_{11}(p) & \tilde{a}_{21}(p)^H\\ \tilde{a}_{21}(p) & \tilde{A}_{22}(p)\end{pmatrix},\quad \tilde{a}_{11}(p)\in\mathbb{R} \tag{1.3}$$

and introduce a vector-valued function

$$f(z, p)=\tilde{a}_{21}(p)-\tilde{a}_{11}(p)z+\tilde{A}_{22}(p)z-z\tilde{a}_{21}(p)^H z, \tag{1.4}$$

where

$$f=(f_1\cdots, f_{n-1})^T,\quad z=(\zeta_1,\cdots,\zeta_{n-1})^T\in\mathbb{C}^{n-1},\quad p\in\mathbb{R}^N.$$

Let

$$f_j=\varphi_j+i\psi_j,\quad \zeta_j=\xi_j+i\eta_j,\quad i=\sqrt{-1},\ j=1, \cdots, n-1$$

and

$$x=(\xi_1, \cdots, \xi_{n-1})^T,\quad y=(\eta_1, \cdots, \eta_{n-1})\in\mathbb{R}^{n-1}.$$

Obviously, $\varphi_j(x, y, p)$ and $\psi_j(x, y, p)$ $(j=1, \cdots, n-1)$ are real analytic functions of real variables $x, y\in\mathbb{R}^{n-1}$ and $p\in\mathscr{B}(0)$, and the functions satisfy

$$\varphi_j(0,0,0)=0,\quad \psi_j(0,0,0)=0,\quad j=1,\cdots,n-1. \tag{1.5}$$

Since $f_1, \cdots, f_{n-1}$ are complex analytic functions of the complex variables $\zeta_1, \cdots, \zeta_{n-1}$ for any $p\in\mathscr{B}(0)$, we have ([1, p. 39, Theorem 8])

$$\det\frac{\partial(\varphi_1,\cdots,\varphi_{n-1},\psi_1,\cdots,\psi_{n-1})}{\partial(\xi_1,\cdots,\xi_{n-1},\eta_1,\cdots,\eta_{n-1})}=\left|\det\frac{\partial(f_1,\cdots,f_{n-1})}{\partial(\zeta_1,\cdots,\zeta_{n-1})}\right|^2.$$

Combining it with

$$\left(\frac{\partial(f_1,\cdots,f_{n-1})}{\partial(\zeta_1,\cdots,\zeta_{n-1})}\right)_{z=0,\ p=0}=\tilde{A}_{22}(0)-\tilde{a}_{11}(0)I=\Sigma_2^T\Sigma_2-\sigma_1^2 I$$

we get

$$\det\left(\frac{\partial(\varphi_1,\cdots,\varphi_{n-1},\psi_1,\cdots,\psi_{n-1})}{\partial(\xi_1,\cdots,\xi_{n-1},\eta_1,\cdots,\eta_{n-1})}\right)_{\substack{x=y=0\\ p=0}}=\prod_{l=2}^{n}(\sigma_l^2-\sigma_1^2)^2\neq 0.$$

Hence by the Implicit Function Theorem (see [2, p. 277] or [4, Theorem 1.2]) the system of equations