A FOURTH ORDER FINITE DIFFERENCE APPROXIMATION TO THE EIGENVALUES OF A CLAMPED PLATE*

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Abstract

In a 21-point finite difference scheme, assign suitable interpolation values to the fictitious node points. The numerical eigenvalues are then of $O(h^2)$ precision. But the corrected value $\hat{\lambda}_h=\lambda_h+\frac{h^2}{6}\lambda_h^{3/2}$ and extrapolation $\hat{\lambda}_h=\frac{4}{3}\lambda_{\frac{h}{2}}-\frac{1}{3}\lambda_h$ can be proved to have $O(h^4)$ precision.

§ 1. Introduction

Consider the following eigenvalue problem of a clamped plate

$$\begin{cases} \Delta^2 u-\lambda u=0, & (x, y)\in\Omega, \\ u=\dfrac{\partial u}{\partial n}=0, & (x, y)\in\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open area in the X–Y plane, $\partial\Omega$ is the boundary of Ω, and $\frac{\partial}{\partial n}$ denotes the outward normal derivatives.

Let

$$S_h=\{(mh, nh)\,|\,m, n \text{ integer}\},$$
$$\Omega_h=\Omega\cap S_h, \quad \partial\Omega_h=\partial\Omega\cap S_h.$$

Let Δ_h and $\Delta_h^\times$ be the well-known 5-point and skewed 5-point difference operators respectively.

In dealing with (1.1) by numerical methods, usually Δ^2 will be approximated by Δ_h^2, the so called 13-point scheme. Thomée[1] proved λ_h is of $O(h^{1/2})$ precision, where λ_h satisfies:

$$\begin{cases} \Delta_h^2 u_h-\lambda_h u_h=0, & (x, y)\in\Omega_h, \\ u_h=0, & (x, y)\in S_h\backslash\Omega_h. \end{cases} \tag{1.2}$$

Using the operator Δ_h^2 to approximate Δ^2 in irregular interior points[2], Kuttler[3] obtained $O(h^2)$ and $O(h^2|\ln h|^{1/2})$ approximations to the eigenvalues and eigenvectors of (1.1) respectively.

In this paper, the biharmonic operator is approximated using the 21-point stencil

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$$M_h u=\frac{1}{3h^4}\begin{bmatrix} & 1 & 1 & 1 & \\ 1 & -2 & -10 & -2 & 1 \\ 1 & -10 & 36 & -10 & 1 \\ 1 & -2 & -10 & -2 & 1 \\ & 1 & 1 & 1 & \end{bmatrix}u.$$

It is easy to see that

$$M_h=\frac{1}{3}\Delta_h^2+\frac{2}{3}\Delta_h\Delta_h^{\times}.$$

If $u\in C^6(R^2)$, by direct evaluation,

$$(M_h-\Delta^2)u=\frac{h^2}{6}\Delta^3 u+O(h^4). \tag{1.3}$$

Lu et al.[4] applied the 21-point scheme to the biharmonic boundary value problem and showed that the error is $O(h^4)$. Here, we generalize his result to the eigenvalue problem. First, we should point out that a biharmonic operator satisfying (1.1) with $u\in C^4$ on $\partial\Omega$ is positive definite, and hence its square operator $\sqrt{\Delta^2}=-\Delta$ exists and is also positive definite. If $u\in D(\Delta^2)$ is an eigenvector of (1.1), then

$$\Delta^3 u=\lambda\Delta u=-\lambda^{3/2}u$$

and (1.3) becomes

$$(M_h-\Delta^2)u=-\frac{h^2}{6}\lambda^{3/2}u+O(h^4). \tag{1.4}$$

Applications of the correction method to the eigenproblems were first introduced by Kuttler[5], who corrected the 9-point scheme of a Laplace operator. In 1984, one of the authors made some extension of the method[6].

§ 2. Correction Method of the 21-point Scheme

Let

$$\Omega_h'=\{P\in\Omega_h \mid |Q-P|\leqslant\sqrt{5}\,h \text{ implies } Q\in\Omega_h\},$$
$$\Omega_h^*=\Omega_h/\Omega_h'.$$

Ω_h' is the set of regular points and Ω_h^* is the set of irregular points.

Suppose $P\in\Omega_h^*$. In order to evaluate $M_h u_h(P)$, values of u_h on some points outside Ω have to be defined. This can be done by interpolation. The simplest way is to interpolate along the grid lines. For example, if $P_1\in\Omega_h^*$ is an irregular point, $P_{-1}\overline{\in}\Omega_h$, $P_0\in\partial\Omega_h$, $P_i\in\Omega_h$ $(i=2,3,4)$, $P_i=P_0+ihe_h$, where e_h is the unit vector of the inward normal, then

$$I_h u_h(P_{-1})=10u_h(P_1)-5u_h(P_2)+\frac{5}{3}u_h(P_3)-\frac{1}{4}u_h(P_4) \tag{2.1}$$

has $O(h^6)$ precision. Similarly,

$$\hat{I}_h u_h(P_{-1})=6u_h(P_1)-2u_h(P_2)+\frac{1}{3}u_h(P_3) \tag{2.2}$$

is easily proved to have $O(h^5)$ precision. Of course, unequally spaced interpolation formulae on a smooth region are also available.