THE EXISTENCE OF THE SOLUTION AND THE GLOBALLY CONVERGENT SHOOTING METHOD FOR A CLASS OF TWO-POINT BOUNDARY VALUE PROBLEMS*

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Abstract

A class of two-point boundary value problems are studied. A new existence theorem of solution is constructively proved and a globally convergent shooting method for it is given.

§1. Introduction

In this paper we give a new existence theorem of solution for the two-point boundary value problem

$$u'_k = g_k(t, u_1, u_2, \cdots, u_n), \quad k=1, 2, \cdots, n, \ 0\le t\le 1, \tag{1.1}$$

$$u_i(0)=c_i(i=i_1, \cdots, i_{n-r}), \quad u_j(1)=d_j(j=j_1, \cdots, j_r), \tag{1.2}$$

$$\{i_1, \cdots, i_{n-r}\}\cap\{j_1, \cdots, j_r\}=\emptyset,$$

where $g_k(t, u_1, \cdots, u_n)$ $(k=1, \cdots, n)$ are Lipschitz continuous on $[0, t_0]*R^n$ $(t_0>1)$ and $\emptyset$ is an empty set. We prove that if in a proper arrangement of $g_1, \cdots, g_n$ and $u_1, \cdots, u_n$, $g_1, \cdots, g_n$ satisfy

$$\overline{\lim_{\substack{\|U\|\to\infty \\ |u_j|<g}}} |g_i(t, u_1, \cdots, u_n)|/|u_i| < +\infty \text{ for any } g>0, \quad i=1, 2, \cdots, n$$

then (1.1)—(1.2) has at least one solution.

Our result is proved by using a shooting function and the generalized Newton homotopy.

Consider the related initial value problem

$$u'_k=\lambda g_k(t, u_1, \cdots, u_n), \quad k=1, \cdots, n, \tag{1.3}$$

$$u_i(0)=c_i(i=i_1, \cdots, i_{n-r}), \quad u_{j_p}(0)=x_p, \tag{1.4}$$

$$p=1, \cdots, r, \ j_p\in\{1, \cdots, n\}\setminus\{i_1, \cdots, i_{n-r}\}$$

and denote the solution of (1.3), (1.4) by $u_k=u_k(t, x, \lambda)(k=1, \cdots, n)$. Then we seek x such that

$$\begin{aligned} F(x, 1) &= (f_1(x, 1), \cdots, f_r(x, 1))^T \\ &= (u_{j_1}(1, x, 1), \cdots, u_{j_r}(1, x, 1))^T-(d_{j_1}, \cdots, d_{j_r})^T=0. \end{aligned} \tag{1.5}$$

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If $x=x^*$ is a root of (1.5), then $u_*(t)=u_*(t, x, 1)$ is a solution of (1.1)—(1.2). Obviously for any solution $u_*(t, x)$ of (1.1)—(1.2) the value $x_p=u_{j_p}(0)$, $j_p\in\{j_1, \cdots, j_r\}$, is a root of (1.5). Thus solving (1.1)—(1.2) is equivalent to solving equations $F(x, 1)=0$. It is clear that the latter is much easier than the former. The shooting methods are just to solve $F(x, 1)=0$ by some iteration. But most of the shooting methods have only local convergence, i.e. only when the initial shooting point x is chosen in the neighborhood of the real shooting point x^*, can the methods be successful (see [1], [2]). Therefore an open problem, which is also difficult in shooting methods, is to enlarge the domain in which initial points can be chosen. Any globally convergent shooting method is thus very significant. In this paper we prove constructively the existence of the solution of $F(x, 1)=0$ and thus give a globally convergent shooting method for solving numerically (1.1)—(1.2).

For more detailed discussion on globally convergent shooting methods, we refer to Watson[3] and Zhang[4].

§ 2. Existence Theorem and Its Proof

Consider the boundary value problem (1.1)—(1.2). Set $E_1=(e_{i_1}^T, \cdots, e_{i_{n-r}}^T)^T$, $\tilde{E}_1=(e_{k_1}^T, \cdots, e_{k_r}^T)^T$, $\{i_1, \cdots, i_{n-r}\}\cap\{k_1, \cdots, k_r\}=\emptyset$ and $E_2=(e_{j_1}^T, \cdots, e_{j_r}^T)^T$, $\tilde{E}_2=(e_{l_1}^T, \cdots, e_{l_{n-r}}^T)^T$, $\{j_1, \cdots, j_r\}\cap\{l_1, \cdots, l_{n-r}\}=\emptyset$. Then (1.1)—(1.2) and (1.3), (1.4) can be written as

$$U'(t)=G(t, U(t)),\ G=(g_1, \cdots, g_n)^T,\ U=(u_1, \cdots, u_n)^T, \tag{2.1}$$

$$E_1U(0)=C,\ E_2U(1)=D,\ C=(c_{i_1}, \cdots, c_{i_{n-r}})^T,\ D=(d_{j_1}, \cdots, d_{j_r})^T, \tag{2.2}$$

and

$$U'(t)=\lambda G(t, U(t)),\quad 0\leqslant\lambda\leqslant 1, \tag{2.3}$$

$$E_1U(0)=C,\quad E_1U(0)=x. \tag{2.4}$$

By the definition of E_1, $\tilde{E}_1$, E_2, $\tilde{E}_2$,

$$E_1^TE_1+\tilde{E}_1^T\tilde{E}_1=E_2^TE_2+\tilde{E}_2^T\tilde{E}_2=I.$$

Throughout we will assume that $G(t, U(t))$ is a Lipschitz continuous function on $[0, t_0]*R^n(t_0>1)$, $D=0$ and $\|x\|=(x_1^2+, \cdots, +x_n^2)^{\frac{1}{2}}$.

Definition 1. *Matrix $B=(b_{ij})$ is called an Ind (Indication) matrix of G if B is defined as follows:*

Given g, if indexes $j_1, \cdots, j_p\in\{1, \cdots, n\}$ satisfy that for any $q>0$ there exists $\alpha(q)$ such that

$$\begin{gathered}\overline{\lim_{\|U\|\to\infty}}|g_i|/|u_i|=\alpha(q)<+\infty,\\ \|u_j\|<q,\\ j=j_1, \cdots, j_p,\end{gathered} \tag{2.5}$$

then let

$$b_{ij}=\begin{cases}1, & j=j_1, \cdots, j_p\\ 0, & j\neq j_1, \cdots, j_p.\end{cases}$$