USING A PREDICTOR–CORRECTOR SCHEME TO COMPUTE NAVIER–STOKES EQUATIONS IN THREE–DIMENSIONAL SPHERICAL COORDINATES*

CHANG QIAN–SHUN (常谦顺)
(Institute of Applied Mathematics, Academia Sinica, Beijing, China)

Abstract

A new predictor–corrector difference scheme is described for solving the time-dependent Navier-Stokes equations in three-dimensional spherical coordinates. A boundary condition for the pressure is deduced by auxiliary velocity. A multigrid algorithm is employed in solving equations of the pressure. An example of application of this scheme is computed and its results are presented.

§ 1. Introduction

In a general numerical scheme for Navier-Stokes equations, velocities are advanced explicitly in time. In such explicit schemes, the time step is restricted by stability conditions. This is more stringent for a smaller Reynolds number and, especially, in spherical coordinates. A predictor–corrector scheme is given in this paper. It saves computational time and keeps properties of the differential equations $(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\ \boldsymbol{u})=0$. The pressure is computed by a Poisson's equation. A boundary condition for the equation is given by means of auxiliary velocities. In order to reduce computational time, a multigrid algorithm is employed in solving Poisson's equation.

In this paper h_1, h_2 and h_3 are step sizes in r, θ and φ direction, respectively. The following notations for difference operators are used:

$$\delta_r f^n_{i,j,k}=\frac{1}{2h_1}(f^n_{i+1,j,k}-f^n_{i-1,j,k}),\qquad \delta_\theta f^n_{i,j,k}=\frac{1}{2r_i h_2}(f^n_{i,j+1,k}-f^n_{i,j-1,k}),$$

$$\delta_\varphi f^n_{i,j,k}=\frac{1}{2r_i\cdot\sin\theta_j\cdot h_3}(f^n_{i,j,k+1}-f^n_{i,j,k-1}),$$

$$\delta^2_r f^n_{i,j,k}=\frac{1}{r_i^2\cdot h_1^2}\left[r^2_{i+1/2}(f^n_{i+1,j,k}-f^n_{i,j,k})-r^2_{i-1/2}(f^n_{i,j,k}-f^n_{i-1,j,k})\right],$$

$$\delta^2_\theta f^n_{i,j,k}=\frac{1}{r_i^2\cdot\sin\theta_j\cdot h_2^2}\left[\sin\theta_{j+1/2}(f^n_{i,j+1,k}-f^n_{i,j,k})-\sin\theta_{j-1/2}(f^n_{i,j,k}-f^n_{i,j-1,k})\right],$$

$$\delta^2_\varphi f^n_{i,j,k}=\frac{1}{r_i^2\sin^2\theta_j\cdot h_3^2}\left[f^n_{i,j,k+1}-2f^n_{i,j,k}+f^n_{i,j,k-1}\right],$$

* Received August 7, 1986.

$$\Delta_\theta f^n_{i,j,k}=\frac{1}{2r_i\cdot\sin\theta_j\cdot h_2}\left[\sin\theta_{j+1/2}(f^n_{i,j+1,k}+f^n_{i,j,k})-\sin\theta_{j-1/2}(f^n_{i,j,k}+f^n_{i,j-1,k})\right].$$

§ 2. Differential Equations and Basic Algorithm

We consider the Navier-Stokes problem in spherical coordinates

$$\begin{aligned}&\frac{\partial u}{\partial t}+\left[u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial\theta}+\frac{w}{r\cdot\sin\theta}\frac{\partial u}{\partial\varphi}-\frac{v^2+w^2}{r}\right]\\&\quad-\nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\cdot\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\cdot\sin^2\theta}\frac{\partial^2u}{\partial\varphi^2}\right.\\&\quad\left.-\frac{2}{r^2}\left(u+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot v)+\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}\right)\right]+\frac{\partial p}{\partial r}=F_1,\end{aligned}\tag{2.1}$$

$$\begin{aligned}&\frac{\partial v}{\partial t}+\left[u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial\theta}+\frac{w}{r\cdot\sin\theta}\frac{\partial v}{\partial\varphi}+\frac{uv}{r}-\frac{w^2}{r}\operatorname{ctg}\theta\right]\\&\quad-\nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{r^2\cdot\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial v}{\partial\theta}\right)+\frac{1}{r^2\cdot\sin^2\theta}\frac{\partial^2v}{\partial\varphi^2}\right.\\&\quad\left.+\frac{2}{r^2}\left(\frac{\partial u}{\partial\theta}-\frac{v}{2\sin^2\theta}-\frac{\cos\theta}{\sin^2\theta}\frac{\partial w}{\partial\varphi}\right)\right]+\frac{1}{r}\frac{\partial p}{\partial\theta}=F_2,\end{aligned}\tag{2.2}$$

$$\begin{aligned}&\frac{\partial w}{\partial t}+\left[u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial\theta}+\frac{w}{r\cdot\sin\theta}\frac{\partial w}{\partial\varphi}+\frac{u\cdot w}{r}+\frac{vw}{r}\operatorname{ctg}\theta\right]\\&\quad-\nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right)+\frac{1}{r^2\cdot\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right)+\frac{1}{r^2\cdot\sin^2\theta}\frac{\partial^2w}{\partial\varphi^2}\right.\\&\quad\left.+\frac{2}{r^2\cdot\sin\theta}\left(\frac{\partial u}{\partial\varphi}+\operatorname{ctg}\theta\frac{\partial v}{\partial\varphi}-\frac{w}{2\cdot\sin\theta}\right)\right]+\frac{1}{r\cdot\sin\theta}\frac{\partial p}{\partial\varphi}=F_3,\end{aligned}\tag{2.3}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u)+\frac{1}{r\cdot\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot v)+\frac{1}{r\cdot\sin\theta}\frac{\partial w}{\partial\varphi}=0,\tag{2.4}$$

$$\boldsymbol{u}|_\Gamma=0,\tag{2.5}$$

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(r\cdot\theta\cdot\varphi),\tag{2.6}$$

where $\boldsymbol{u}=(u,v,w)$ is the velocity vector, p is the ratio of the pressure to constant density (for brevity, we refer to p simply as pressure) and ν is a kinematic viscosity coefficient.

In order to deduce the basic splitting scheme and the boundary condition of the pressure, we write the Navier-Stokes problem in vector form:

$$\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p=\nu\Delta\boldsymbol{u}+\boldsymbol{F},\tag{2.7}$$

$$\nabla\cdot\boldsymbol{u}=0,\tag{2.8}$$

$$\boldsymbol{u}|_\Gamma=0,\tag{2.9}$$

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(r\cdot\theta\cdot\varphi).\tag{2.10}$$

First, equations (2.7)—(2.10) are written in difference form only for time:

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\tau}=-\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n-\nabla p^{n+1}+\nu\Delta\boldsymbol{u}^n+\boldsymbol{F}^n,\tag{2.11}$$