SENSITIVITY ANALYSIS OF ZERO SINGULAR VALUES AND MULTIPLE SINGULAR VALUES*1)

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Abstract

Some results of the author[3,4] are used to discuss the sensitivity of zero singular values and multiple singular values of a real matrix analytically dependent on several parameters.

§ 1. Preliminaries

Let $\mathbb{R}^{m\times n}$ denote the set of real $m\times n$ matrices, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$ and

$$S\mathbb{R}^{n\times n}=\{A\in\mathbb{R}^{n\times n}:\ A^T=A\},$$

in which the superscript T is for transpose. The singular values of a matrix $A\in\mathbb{R}^{m\times n}$ are denoted by $\sigma_1(A), \cdots, \sigma_n(A)$, the eigenvalues of a matrix $A\in\mathbb{R}^{n\times n}$ are denoted by $\lambda_1(A), \cdots, \lambda_n(A)$, and

$$\sigma(A)=\{\sigma_j(A)\}_{j=1}^n,\ \ \lambda(A)=\{\lambda_j(A)\}_{j=1}^n.$$

The symbol $\| \ \|_2$ denotes the usual Euclidean vector norm and the spectral norm.

Let $p=(p_1, \cdots, p_N)^T\in\mathbb{R}^N$. Suppose that $A(p)\in\mathbb{R}^{m\times n}$ is a real matrix-valued analytic function in some open set $\mathscr{S}\subset\mathbb{R}^N$. It is well known that for any point $p^*\in\mathscr{S}$, if $\sigma^*>0$ is a simple singular value of $A(p^*)$, then there is a real analytic function $\sigma(p)>0$ defined in some neighbourhood $\mathscr{B}(p^*)\subset\mathscr{S}$ of p^* such that $\sigma(p^*)=\sigma^*$, $\sigma(p)\in\sigma(A(p))$ $\forall p\in\mathscr{B}(p^*)$, and one can obtain perturbation expansions of $\sigma(p)$ at the point p^* (see [1], [5]). In such a case we may define the sensitivity of the singular value σ^* with respect to the parameter p_j by

$$s_{p_j}(\sigma^*)=\left|\left(\frac{\partial\sigma(p)}{\partial p_j}\right)_{p=p^*}\right|.$$

But if σ^* is a zero singular value or a multiple singular value of $A(p^*)$, then, in general, there is no real differentiable function $\sigma(p)\geqslant 0$ defined in some neighbourhood $\mathscr{B}(p^*)\subset\mathscr{S}$ of p^* satisfying $\sigma(p^*)=\sigma^*$ such that $\sigma(p)\in\sigma(A(p))$ $\forall p\in\mathscr{B}(p^*)$. This paper will discuss the sensitivity of σ^* with respect to p_j in these cases.

Without loss of generality we may assume that the point p^* is the origin of $\mathbb{R}^N$ and $p^*\in\mathscr{S}$. For any real-valued function $f(p)$ defined in $\mathscr{S}$, we shall denote the right and left partial derivatives of $f(p)$ with respect to p_j at the origin, respectively, by

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$$\left(\frac{\partial f(p)}{\partial p_j}\right)_{p=0,\ p_j=+0}=\lim_{p_j\to+0}\frac{f(0,\ \cdots,\ 0,\ p_j,\ 0,\ \cdots,\ 0)-f(0,\ \cdots,\ 0)}{p_j}$$

and

$$\left(\frac{\partial f(p)}{\partial p_j}\right)_{p=0,\ p_j=-0}=\lim_{p_j\to-0}\frac{f(0,\ \cdots,\ 0,\ p_j,\ 0,\ \cdots,\ 0)-f(0,\ \cdots,\ 0)}{p_j},\ j=1,\ \cdots,\ N.$$

Now we cite a result about partial derivatives of eigenvalues of a real symmetric matrix (see [3, Theorems 2.3 and 2.4], and [4, Theorem 2.1]), on the basis of which we may discuss the sensitivity of zero singular values and multiple singular values.

Theorem 1.1. *Let $p=(p_1,\ \cdots,\ p_N)^T\in\mathbb{R}^N$, and let $A(p)\in S\mathbb{R}^{n\times n}$ be a real analytic function in some neighbourhood $\mathscr{B}(0)\subset\mathbb{R}^N$ of the origin. Suppose that λ_1 is an eigenvalue of $A(0)$ with multiplicity $r\geqslant1$, i.e., there is a real orthogonal matrix $X\in\mathbb{R}^{n\times n}$ such that*

$$X=(\underset{r}{X_1},\ \underset{n-r}{X_2}),\ \ X^TA(0)X=\begin{pmatrix}\lambda_1I^{(r)} & 0\\ 0 & A_2\end{pmatrix},\ \ \lambda_1\overline{\in}\lambda(A_2)\tag{1.1}$$

(in the case of $r=1$, we rewrite $X_1=x_1$). Then

(i) *if $r=1$, there is a real analytic function $\lambda_1(p)$ defined in some neighbourhood $\mathscr{B}_0\subset\mathscr{B}(0)$ of the origin such that $\lambda_1(0)=\lambda_1$, $\lambda_1(p)\in\lambda(A(p))$, $\forall p\in\mathscr{B}_0$, and one has*

$$\left(\frac{\partial\lambda_1(p)}{\partial p_j}\right)_{p=0}=x_1^T\left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}x_1,\quad j=1,\cdots,N\tag{1.2}$$

and

$$\begin{aligned}\left(\frac{\partial^2\lambda_1(p)}{\partial p_j\partial p_k}\right)_{p=0}&=x_1^T\left(\frac{\partial^2A(p)}{\partial p_j\partial p_k}\right)_{p=0}x_1\\&\quad+2x_1^T\left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}X_2(\lambda_1I-A_2)^{-1}X_2^T\left(\frac{\partial A(p)}{\partial p_k}\right)_{p=0}x_1,\\&\qquad j,\ k=1,\ \cdots,\ N;\end{aligned}\tag{1.3}$$

(ii) *if $r>1$, there are real-valued functions $\lambda_1(p),\ \cdots,\ \lambda_r(p)$ defined in some neighbourhood $\mathscr{B}_0\subset\mathscr{B}(0)$ of the origin and two permutations π and π' of $1,\ \cdots,\ r$ such that*

$$\lambda_1(0)=\cdots=\lambda_r(0)=\lambda_1,\ \lambda_1(p),\ \cdots,\ \lambda_r(p)\in\lambda(A(p))\quad\forall p\in\mathscr{B}_0,$$

and one has

$$\left(\frac{\partial\lambda_s(p)}{\partial p_j}\right)_{p=0,\ p_j=+0}=\lambda_{\pi(s)}\left(X_1^T\left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}X_1\right)\tag{1.4}$$

and

$$\left(\frac{\partial\lambda_s(p)}{\partial p_j}\right)_{p=0,\ p_j=-0}=\lambda_{\pi'(s)}\left(X_1^T\left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}X_1\right),\ s=1,\ \cdots,\ r,\ j=1,\ \cdots,\ N.\tag{1.5}$$

We shall prove some formulas for partial derivatives in Section 2 and give numerical examples in Section 3.