ON S–STABILITY *

ZHAO SHUANG–SUO DONG GUO–XIONG QIAO NING
(*Lanzhou University, Lanzhou, Gansu*)

Abstract

We prove in this paper that no consistent and well–defined Runge–Kutta method is S–stable and point out the errors of the theorems on S–stability in [1].

1. Introduction

To further study the stability of a general R–K method

$$y_{n+1} = y_n + \sum_{i=1}^{r} b_i k_i, \quad k_i = hf(t_n + c_i h, y_n + \sum_{j=1}^{r} a_{ij} k_j), \quad i = 1(1)r, \tag{1.1}$$

which is used to solve a stiff initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0, \quad y_0, y, f \in R^N, t_0 < t \le T, \tag{1.2}$$

A. Prothero and A. Robinson presented in [1] the concepts of S–stability and strong S–stability, and derived necessary and sufficient conditions for both stabilities (Theorems 2.1 and 2.2 in [1]). Then they discussed stabilities of several classes of well–defined and consistent R–K methods and concluded that these methods are S–stable or strongly S–stable.

Their work has a great influence on the research of numerical methods of stiff O. D. E.. The concepts and theorems of S–stability and strong S–stability have been adopted by many authors (see [2]–[7]).

Based on the definition of S–stability in [1], we now prove that consistent and well–defined R–K methods are not S–stable, and therefore not strongly S–stable. Then we point out the errors in Theorems 2.1 and 2.2 in [1].

For convenience, here we introduce briefly the definitions and some main conclusions of S–stability and strong S–stability in [1] and adopt the symbols of [1] as much as we can.

* Received October 22, 1986.

2. Definition of S–Stability and Some Main Conclusions in [1]

Definition 2.1. *A R–K method (1.1) is said to be S-stable if it is applied to the test equation*

$$y' = \lambda(y - g(t)) + g'(t), \qquad g \in G \tag{2.1}$$

(where λ is a complex constant with $\mathrm{Re}(\lambda) < 0$, and G is the set of all functions defined in $[t_0, T]$, which have first bounded–derivative), and for any real positive constant λ_0 and any $g(t) \in G$, there exists a real positive constant h_0, such that

$$|\varepsilon_{n+1}| < |\varepsilon_n|, \ \ \forall h \in (0, h_0), \ \ \forall \lambda \text{ with } \mathrm{Re}(-\lambda) \geq \lambda_0, \ \ t_n, t_{n+1} \in [t_0, T] \tag{2.2}$$

provided $y_n \neq g(t_n)$, where $\varepsilon_n = y_n - g(t_n)$.

Furthermore, (1.1) is said to be strongly S–stable if it is S–stable and

$$\varepsilon_{n+1}/\varepsilon_n \to 0, \ \ \forall h \in (0, h_0), \text{ as } \mathrm{Re}(-\lambda) \to \infty, \ \ t_n, t_{n+1} \in [t_0, T]. \tag{2.3}$$

Since the solution of (2.1) is $y(t) = g(t) + (y_0 - g(t_0))e^{\lambda(t-t_0)}$ and $g(t)$ is quite arbitrary, the methods with S–stability and strong S–stability are very satisfactory. That is why many authors studied the construction of S–stable and strongly S–stable methods.

Correspondingly to [1], note $z = 1/(\lambda h)$. Applying (1.1) to (2.1), we obtain

$$\varepsilon_{n+1} = \alpha(z)\varepsilon_n + h\beta(z),$$

where

$$\begin{cases} \alpha(z) = 1 - b^T(A - zI)^{-1}e, \ \ A = (a_{ij}), \\ e = (1, 1, \cdots, 1)^T, \ \ b = (b_1, \cdots, b_r)^T, \\ \beta(z) = -G_0 + b^T(A - zI)^{-1}(\frac{1}{h}(\tilde{g} - g(t_n)e) - z\tilde{g}'), \\ G_0 = (g(t_{n+1}) - g(t_n))/h, \\ \tilde{g} = (g(t_n + c_1 h), \cdots, g(t_n + c_r h))^T, \\ \tilde{g}' = (g'(t_n + c_1 h), \cdots, g'(t_n + c_r h))^T. \end{cases} \tag{2.4}$$

Lemma 2.1. *Assume $R = \{z | 0 < \mathrm{Re}(-z) \leq \bar{z}\}$ and $\bar{z}$ is a real positive number. Define*

$$\varepsilon(z, h, \varepsilon_0) = \alpha(z)\varepsilon_0 + h\beta(z), \ \ \forall \varepsilon_0 \in C, \ \ \forall h \in (0, \bar{h}), \ \ \forall z \in R, \tag{2.5}$$

where $\bar{h}$ is a real positive number . Then for any $g \in G$, there exists a real positive number $h_0 = h_0(\bar{z}, \varepsilon_0) \leq \bar{h}$, such that

$$|\varepsilon(z, h, \varepsilon_0)| < |\varepsilon_0|, \ \ \forall \varepsilon_0 \neq 0, \ \ \forall h \in (0, h_0), \ \ \forall z \in R$$

if and only if