ITERATIVE METHODS FOR BOUNDARY INTEGRAL EQUATIONS

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Abstract

We review some iterative methods for solving boundary integral equations which arise in Dirichlet and Neumann problems for the Helmholtz and Laplace equations. In particular we show how these integral equations may be transformed so that they may be solved by Neumann-Poincare Picard iteration.

1. Introduction

We review some iterative methods for solving boundary integral equations which are simple, effective, and largely overlooked by engineers and scientists interested in numerical solutions of problems of practical interest. The boundary integral equations considered are representative of a wide class which may be solved by iteration, however here we will restrict attention to these arising in the Dirichlet and Neumann problems for the Laplace and Helmholtz equations.

The boundary value problems consist of finding solutions at the partial differential equation $(\nabla^2 + k^2)u = 0$ for points p either interior or exterior to a smooth closed bounded simply connected surface Γ in $I\!R^3$ with exterior Ω_+ and interior Ω_- which take on boundary values $u|_\Gamma = f$ (Dirichlet) or $\left.\frac{\partial u}{\partial n}\right|_\Gamma = f$ (Neumann). Included is the case when $k = 0$ where Laplace's equation is the governing field equation. In addition, considering the exterior problem, we impose a radiation condition when $k \neq 0, r(\frac{du}{\partial r} - iku) = o(1)$ uniformly in (θ, ϕ) as $r \to \infty$ where (r, θ, ϕ) are the spherical coordinates of a point p in Ω_+, and a regularity condition when $k = 0, u = O(\frac{1}{r})$ as $r \to \infty$.

As is well known these problems may be cast as boundary integral equations. Using the notation and results of Kleinman and Roach [11] we define single and double layer operators by

$$S\omega := \int_\Gamma \omega(q)\gamma(p,q)\, ds_q, \quad p \in I\!R^3, \tag{1}$$

$$D\omega := \int_\Gamma \omega(q) \frac{\partial}{\partial nq} \gamma(p,q)\, ds_q, \quad p \notin \Gamma, \tag{2}$$

where

$$\gamma(p,q) = -\frac{e^{ik|p-q|}}{2\pi|p-q|} \tag{3}$$

and boundary integral operators

$$K\omega = \int_\Gamma \omega(q) \frac{\partial}{\partial n_p} \gamma(p,q) ds_q, \quad p \in \Gamma, \tag{4}$$

$$\overline{K}^*_\omega = \int_\Gamma \omega(q) \frac{\partial}{\partial n_q} \gamma(p,q) ds_q, \quad p \in \Gamma, \tag{5}$$

$$\frac{\partial}{\partial n} D\omega = \frac{\partial}{\partial n_p} \int_\Gamma \omega(q) \frac{\partial}{\partial n_q} \gamma(p,q) ds_q, \quad p \in \Gamma. \tag{6}$$

Then the solution of the Dirichlet problem using the layer ansatz is

$$u = \mp D\omega, \quad \begin{matrix} + & p \in \Omega_+, \\ - & p \in \Omega_-, \end{matrix} \tag{7}$$

where ω is a solution of the boundary integral equation

$$\omega \mp \overline{K}^* \omega = f, \quad \begin{matrix} - \text{ exterior Dirichlet problem ,} \\ + \text{ interior Dirichlet problem} \end{matrix} \tag{8}$$

whereas using Green's theorem, the so called direct method, the solution is given by

$$u = \mp \frac{1}{2}(Df - S\omega), \quad \begin{matrix} - & p \in \Omega_+, \\ + & p \in \Omega_-, \end{matrix} \tag{9}$$

where

$$\omega \mp K\omega = \mp D_n f, \quad \begin{matrix} - \text{ exterior Dirichlet problem ,} \\ + \text{ interior Dirichlet problem .} \end{matrix} \tag{10}$$

Similarly the solution of the Neumann problem using layers is given by

$$u = \pm S\omega, \quad \begin{matrix} +, & p \in \Omega_+, \\ -, & p \in \Omega_-, \end{matrix} \tag{11}$$

where

$$\omega \pm K\omega = f, \quad \begin{matrix} + \text{ exterior Neumann problem ,} \\ - \text{ interior Neumann problem} \end{matrix} \tag{12}$$