FINITE DIMENSIONAL APPROXIMATION OF BRANCHES OF SOLUTIONS OF NONLINEAR PROBLEMS NEAR A CUSP POINT*

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Abstract

This paper presents some results on finite dimensional approximation of branches of solutions of nonlinear problems near a cusp point. These results can be applied to numerical methods of solving nonlinear differential equations.

1. Introduction

Consider nonlinear problems of the form

$$F(\lambda, u) = 0$$

where F is a sufficiently smooth function from $R \times V$ into V for some Banach space V. In [1]–[3], finite dimensional approximation of branches of solutions near a simple limit point and a simple bifurcation point were studied respectively. We will consider here the finite dimensional approximation of branches of solutions of problem (1.1) near a cusp point and obtain results similar to that of [3].

Section 2 is devoted to general analysis of the cusp point of branches of solutions of nonlinear problems. In Section 3 we discuss the finite dimensional approximation of branches of solutions near a cusp point of problem (1.1). In Section 4 we apply our results to the Galerkin approximations of nonlinear problems.

2. Local Analysis of the Continuous Problem Near a Cusp Point

Let V, W be real Banach spaces with the norm $\|\cdot\|_V$ and $\|\cdot\|_W$ respectively and G be a C^p mapping from $R \times V$ into $W (p \geq 4)$ and T be a linear compact operator from W into V. We set

$$F(\lambda, u) = u + TG(\lambda, u). \tag{2.1}$$

* Received December 25, 1986.

We assume that $(\lambda_0, u_0) \in R \times V$ is a simple critical point of F in the sense that

(i) $F^0 \equiv (\lambda_0, u_0) = 0$;

(ii) $D_u F^0 \equiv D_u F(\lambda_0, u_0) = I + TD_u G(\lambda_0, u_0) \in \mathcal{L}(V; V)$ is singular and -1 is an eigenvalue of the compact operator $TD_u G(\lambda_0, u_0)$ with the algebraic multiplicity 1;

(iii) $D_\lambda F^0 \equiv D_\lambda F(\lambda_0, u_0) \in$ Range $(D_u F^0)$.

We want to solve the equation

$$F(\lambda, u) = 0 \tag{2.3}$$

in a neighborhood of the simple critical point (λ_0, u_0).

As a consequence of (2.2) (ii) and the theory of linear operators; there exists $\varphi_0 \in V$ such that

$$\begin{aligned} &D_u F^0 \cdot \varphi_0 = 0, \quad \|\varphi_0\|_V = 1, \\ &V_1 \equiv \text{Ker}(D_u F^0) = R \cdot \varphi_0. \end{aligned} \tag{2.4}$$

We denote by V' the dual space of V and by $< \cdot, \cdot >$ the duality pairing between the spaces V and V'. Then there exists $\varphi_0^* \in V'$ such that

$$\begin{aligned} &(D_u F^0)^* \cdot \varphi_0^* = 0, \quad < \varphi_0, \varphi_0^* >= 1, \\ &V_2 \equiv \text{Range } (D_u F^0) = \{v \in V; < v, \varphi_0^* >= 0\}. \end{aligned} \tag{2.5}$$

Finally, we have

$$V = V_1 \oplus V_2$$

and $D_u F^0$ is an isomorphism of V_2. We denote by $L = (D_u F^0|_{V_2})^{-1} \in \mathcal{L}(V_2; V_2)$ the inverse isomorphism of $D_u F^0|_{V_2}$.

Let us define the projection operator $Q : V \to V_2$ by

$$Qv = v - \langle v, \varphi_0^* \rangle \varphi_0, \qquad \forall v \in V. \tag{2.6}$$

Then Eq. (2.3) is equivalent to the system

$$\begin{aligned} &QF(\lambda, u) = 0, \\ &(I - Q)F(\lambda, u) = 0. \end{aligned} \tag{2.7}$$

By the implicit function theorem, there exist two positive constants ξ_0, α_0 and a unique C^p mapping $V : [-\xi_0, \xi_0] \times [-\alpha_0, \alpha_0] \to V_2$ such that

$$\begin{aligned} &QF(\lambda_0 + \xi, u_0 + \alpha\varphi_0 + v(\xi, \alpha)) = 0, \\ &v(0, 0) = 0. \end{aligned} \tag{2.8}$$