A NEW ALGORITHM FOR THE EIGENVALUE PROBLEM OF MATRICES*

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Abstract

In this paper, a new algorithm for the eigenvalue problem of matrices is given. Namerical examples show that it could be a remarkable approach for practical purposes. Some open problems are listed.

§1. Introduction

The prevalent technique for obtaining the eigenvalue of matrices are based on either (a) a factorisation of the matrix A into special factors (say LU, QR) leading to a matrix sequence $\{A_k\}$, which is isospectral with the matrix A and, in a sense, tends to some limit while k goes to infinity, or (b) Jacobi-like methods, power method and others.

To solve a system of linear equations, a new factorization and splitting procedure (QIF) is proposed in [1], which is more convenient for parallel computation. Recently, a more detailed analysis for this factorization is given in [2].

Based on QIF, an algorithm for the eigenvalue problem of matrices is given in this paper, which is essentially a block LU facforization. In §2, the QIF procedure is briefly introduced. §3 is devoted to the algorithm description. The proof of convergence is given in §4. Several numerical examples are presented in §5. They show that the algorithm here could be an attractive method. Finally, some open problems are listed in §6.

§2. QIF Factorization

Let A be an $n \times n$ matrix. Now we consider a factorization of the form:

$$A = WZ \tag{2.1}$$

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where W and Z have the matrix form:

$$W = \begin{bmatrix} 1 & & \mathbf{0} & 0 \\ W_{2,1} & 1 & & W_{2,n} \\ \vdots & W_{3,2} & \mathbf{1} & \vdots \\ W_{n-1,1} & & \mathbf{0} & W_{n-1,n} \\ 0 & & & 1 \end{bmatrix}, Z = \begin{bmatrix} z_{1,1} & z_{1,2} & \cdots & & z_{1,n} \\ & z_{2,2} & & z_{2n-1} & \\ \mathbf{0} & & z_{i,i} & & \mathbf{0} \\ & & & & \\ z_{n,1} & & & & z_{n,n} \end{bmatrix} \tag{2.2}$$

where the elements of W and Z are given by

$$w_{i,j} = \begin{cases} 1, & i = j, \\ w_{i,j}, & (i,j) \in D, \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

and

$$z_{i,j} = \begin{cases} 0, & (i,j) \in D, \\ z_{i,j}, & \text{otherwise}, \end{cases} \tag{2.4}$$

$$D = \{(i,j) | (i = 2, \cdots, [(n+1)/2] \text{ and } j = i-1) \text{ or}$$
$$(i = [(n+1)/2], \cdots, (n-1) \text{ and } j = i+1)\}. \tag{2.5}$$

By substituting (2.2)–(2.3) into (2.1) and comparing corresponding terms of the matrices A and WZ, we have

(i) The elements of the first and last rows of Z are given immediately by

$$z_{1,i} = a_{1,i} \quad \text{and} \quad z_{n,i} = a_{n,i}, \quad i = 1, 2, \cdots, n.$$

(2) Then the sets of $n \times n$ linear systems given by:

$$\begin{aligned} z_{1,1} w_{i,1} + z_{n,1} w_{i,n} &= a_{i,1}, \\ z_{1,n} w_{i,1} + z_{n,n} w_{i,n} &= a_{i,n} \end{aligned} \tag{2.6}$$

are solved to get the values of $w_{i,1}$ and $w_{i,n}$ for $i = 1, 2, \cdots, n-1$. This completes the first stage and calculation of the outermost ring of matrices W and Z. The remaining elements of W and Z are computed in a similar way. Totally $(n-1)/2$ such steps are needed to compute matrices W and Z.

In [2], a necessary and sufficient condition for this procedure without permutations is presented. A pivot strategy is discussed there.

§3. QIF Algorithm Description

Let $A_1 = A$. The algorithm is similar to the QR method except that QR transformation is replaced by the QIF factorisation method in §2.