THE ALGEBRAIC PERTURBATION METHOD FOR GENERALIZED INVERSES*

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1. Introduction

Algebraic perturbation methods were first proposed for the solution of nonsingular linear systems by R. E. Lynch and T. J. Aird [2]. Since then, the algebraic perturbation methods for generalized inverses have been discussed by many scholars [3]–[6]. In [4], a singular square matrix was perturbed algebraically to obtain a nonsingular matrix, resulting in the algebraic perturbation method for the Moore-Penrose generalized inverse. In [5], some results on the relations between nonsingular perturbations and generalized inverses of $m \times n$ matrices were obtained, which generalized the results in [4]. For the Drazin generalized inverse, the author has derived an algebraic perturbation method in [6].

In this paper, we will discuss the algebraic perturbation method for generalized inverses with prescribed range and null space, which generalizes the results in [5] and [6].

We remark that the algebraic perturbation methods for generalized inverses are quite useful. The applications can be found in [5] and [8].

In this paper, we use the same terms and notations as in [1].

2. Main Results

First, we will give two lemmas.

Lemma 1. *Let $A \in C_r^{n\times n}$, and let L and K be subspaces of C^n of dimension $s \le r$ and $n-s$ respectively. $AL \oplus K = C^n, B$ and $C^* \in C_{n-s}^{n\times(n-s)}$ are matrices whose columns form bases for K and $L^{\perp}$ respectively. Then*

$$\begin{bmatrix} T & B \\ C & 0 \end{bmatrix}$$

is nonsingular, and

$$\begin{bmatrix} T & B \\ C & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A_{L,K}^{(2)} & P_{(A^*K^{\perp})^{\perp},L}C^+ \\ B^+P_{K,AL} & -I_{n-s} \end{bmatrix}$$

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where $T = A + BC - AP_{(A^*K^\perp)^\perp,L}$.

Proof. It is easy to show that

$$AL \oplus K = C^n \iff (A^*K^\perp)^\perp \oplus L = C^n \qquad \text{(see [7])}$$

so that $P_{K,AL}, P_{(A^*K^\perp)^\perp,L}$ and $A^{(2)}_{L,K}$ exist.

From $L = N(C)$, it follows that

$$CA^{(2)}_{L,K} = 0, \quad CP_{L,(A^*K^\perp)^\perp} = 0 \tag{1}$$

and

$$\begin{aligned} TP_{(A^*K^\perp)^\perp,L}C^+ - B &= (A + BC - AP_{(A^*K^\perp)^\perp,L})P_{(A^*K^\perp)^\perp,L}C^+ - B \\ &= BCP_{(A^*K^\perp)^\perp,L}C^+ - B \\ &= BCC^+ - B = 0 \end{aligned} \tag{2}$$

and

$$CP_{(A^*K^\perp)^\perp,L}C^+ = CC^+ = I_{n-s}. \tag{3}$$

Finally, obviously $BB^+ = P_{R(B)} = P_K$, and $BB^+P_{K,AL} = P_{K,AL}$ so that

$$\begin{aligned} TA^{(2)}_{L,K} + BB^+P_{K,AL} &= (A + BC - AP_{(A^*K^\perp)^\perp,L})A^{(2)}_{L,K} + P_{K,AL} \\ &= AA^{(2)}_{L,K} + P_{K,AL} \\ &= P_{AL,K} + P_{K,AL} = I_n. \end{aligned} \tag{4}$$

Since $R(AA^{(2)}_{L,K}) = AL$ and $N(AA^{(2)}_{L,K}) = K$. From (1)–(4), we have

$$\begin{bmatrix} T & B \\ C & 0 \end{bmatrix} \cdot \begin{bmatrix} A^{(2)}_{L,K} & P_{(A^*K^\perp)^\perp,L}C^+ \\ B^+P_{K,AL} & -I_{n-s} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n-s} \end{bmatrix}$$

which is the required result.

Lemma 2. *Let* $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ *be a partitioned matrix which is nonsingular, and let the submatrix* A_{22} *also be nonsingular. Then*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}_{11,2} & -A^{-1}_{11,2}A_{12}A^{-1}_{22} \\ -A^{-1}_{22}A_{21}A^{-1}_{11,2} & A^{-1}_{22} + A^{-1}_{22}A_{21}A^{-1}_{11,2}A_{12}A^{-1}_{22} \end{bmatrix}$$

where $A_{11,2} = A_{11} - A_{12}A^{-1}_{22}A_{21}$.

Theorem 1. *Let* $A \in C^{m\times n}_r$. *L is a subspace of C^n of dimension $s \le r$, and K is a subspace of C^m of dimension $m - s$. Suppose $AL \oplus K = C^m$, and $B \in C^{m\times(m-s)}_{m-s}, C^* \in C^{n\times(n-s)}_{n-s}$ are matrices whose columns form bases for K and $L^\perp$ respectively. If $m = n$, let $T = A + BC - AP_{(A^*K^\perp)^\perp,L}$. If $m > n$, let $B = [B_1 \vdots B_2]$ where $B_1 \in C^{m\times(n-s)}_{n-s}$, and*