ON THE SOLUTION OF A CLASS OF TOEPLITZ SYSTEMS*

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Abstract

The solution of certain Toeplitz linear systems is considered in this paper. This kind of systems are encountered when we solve certain partial differential equations by finite difference techniques and approximate functions using higher order splines. The methods presented here are more efficient than the Cholesky decomposition method and are based on the circulant factorisation of the symmetric "banded circulant" matrix, the Woodbury formula and the algebraic perturbation method.

1. Introduction

We consider a linear system of the form

$$Ax = f, \tag{1.1}$$

where the coefficient matrix is an nth order symmetric banded matrix of Toeplitz form

$$A_t = \begin{bmatrix} \alpha_0 & \alpha_1 & & \cdots & \alpha_p & & & & \\ \alpha_1 & \ddots & \ddots & & \ddots & & & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & & & \\ \alpha_p & & \ddots & \ddots & \ddots & & \ddots & & \\ & \ddots & & \ddots & \ddots & \ddots & & \ddots & \\ & & \ddots & & \ddots & \ddots & \ddots & & \alpha_p \\ & & & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & & \ddots & \ddots & \alpha_1 \\ & & & & & \alpha_p & \cdots & \alpha_1 & \alpha_0 \end{bmatrix}, \tag{1.2}$$

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or the symmetric "banded circulant" form

$$A_c = \begin{bmatrix} \alpha_0 & \alpha_1 & & \cdots & \alpha_p & & \alpha_p & \cdots & \alpha_1 \\ \alpha_1 & \ddots & \ddots & & \ddots & & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & & & \alpha_p \\ \alpha_p & & \ddots & \ddots & \ddots & & \ddots & & \\ & \ddots & & \ddots & \ddots & \ddots & & \ddots & \\ & & \ddots & & \ddots & \ddots & \ddots & & \alpha_p \\ \alpha_p & & & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & & & \ddots & & \ddots & \ddots & \alpha_1 \\ \alpha_1 & \cdots & \alpha_p & & & \alpha_p & \cdots & \alpha_1 & \alpha_0 \end{bmatrix}, \tag{1.3}$$

$x = (x_1, x_2, \cdots, x_n)^T$ is the unknown n-vector, and f is the given right-hand side.

This class of linear systems occurs in solving a certain kind of boundary value problems by finite difference techniques, in solving biharmonic equations by the Fourier method, and in higher order spline approximation [2, 3, 4, 5, 6, 11].

System (1.1) with coefficient matrix of form (1.2) can be solved by band Cholesky decomposition [7] or by Toeplitz factorization [6]. Although the operation counts of the two methods are about the same, the latter requires less storage. If the system has a coefficient matrix of form (1.3), then the Cholesky decomposition is expensive, and the circulant factorization presented here is more favorable in terms of not only arithmetic operations but also storage requirements. The methods presented in this paper are based on the fact that under certain conditions the matrix in (1.3) can be factored into two simpler circulant matrices, and the corresponding circulant system may then be solved by using the Woodbury formula [8]. Furthermore, the banded Toeplitz matrix may be treated as a perturbation of a circulant matrix, and Toeplitz systems can be solved by the combination of the circulant factorization and algebraic perturbation method [9].

In §2, we will describe the method for factoring a symmetric banded circulant matrix into two circulant matrices. This factorization was used to solve the band circulant system in [3]. The methods for solving band Toeplitz systems will be studied in §3, and finally, some numerical results will be given in §4.

2. Factorization of Banded Circulant Matrices

To factor the banded circulant matrix given by (1.3) we consider the real function with the elements of the matrix as its coefficients

$$\Phi(z) = \alpha_p z^p + \cdots + \alpha_1 z + \alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_p z^{-p}, \tag{2.1}$$

the characteristic function of matrix A_c. Assume, without loss of generality, that $\alpha_p = 1$. We have that following theorem.