MAX-NORM ESTIMATES FOR GALERKIN APPROXIMATIONS OF ONE-DIMENSIONAL ELLIPTIC, PARABOLIC AND HYPERBOLIC PROBLEMS WITH MIXED BOUNDARY CONDITIONS*1)

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Abstract

The Galerkin methods are studied for two-point boundary value problems and the related one-dimensional parabolic and hyperbolic problems. The boundary value problem considered here is of non-adjoint form and with mixed boundary conditions. The optimal order error estimate in the max-norm is first derived for the boundary problem for the finite element subspace $M \subset S_{k+1,s+1}(I)$ with $0 \le k \le s$. This result then gives optimal order max-norm error estimates for the continuous and discrete time approximations for the evolution problems described above.

§1. Introduction

Galerkin methods for the two-point boundary value problems with Dirichlet boundary have been studied intensively in [2], [3], [4], [7], etc. and a series of significant results have been achieved. In this paper, our emphasis is on the boundary condition of mixed-type. In Section 2 an optimal order L^∞ estimate for Galerkin approximations is derived. This result is then applied in Sections 3 and 4 to the single space variable parabolic and hyperbolic equations, respectively, to get the optimal order L^∞ estimates for continuous and discrete time Galerkin approximations.

Consider the following boundary value problems

$$\begin{aligned} &Lu \equiv -(a(x)u')' + b(x)u' + d(x)u = f(x), \quad x \in I = (0,1), \\ &a(0)u'(0) - \sigma_0 u(0) = 0, \quad a(1)u'(1) + \sigma_1 u(1) = 0; \end{aligned} \tag{1.1}$$

and the initial-boundary value problems

$$\begin{aligned} &\frac{\partial u}{\partial t} + Lu = f_1(x,t), \quad (x,t) \in I \times (o,T], \\ &a(0)u'(0) - \sigma_0 u(0) = 0, \quad a(1)u'(1) + \sigma_1 u(1) = 0, \\ &u(x,0) = u_0(x), \quad x \in I, \end{aligned} \tag{1.2}$$

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and

$$\frac{\partial^2 u}{\partial t^2} + Lu = f_2(x,t), \quad (x,t) \in I \times (0,T],$$
$$a(0)u'(0) - \sigma_0 u(0) = 0, \quad a(1)u'(1) + \sigma_1 u(1) = 0, \tag{1.3}$$
$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in I.$$

For problem (1.1), assume that

(i) $a(x) \in C^1(I), b(x) \in C^0(I)$ and $b'(x), d(x) \in L^\infty(I)$;

(ii) $\sigma_0, \sigma_1 \geq 0$ with $\sigma_0^2 + \sigma_1^2 > 0$ and there exist constants $a_0, a_1 > 0$ such that

$$0 < a_0 \leq a(x) \leq a_1, \quad \forall x \in I; \tag{1.4}$$

(iii) for each $f \in L^2(I)$, Problem (1.1) has a unique solution $u(x)$.

Problem (1.1) can be posed as

$$B(u,v) = (f,v), \quad \forall v \in H^1(I), \tag{1.5}$$

where

$$B(\phi,\psi) = (a\phi',\psi') + (b\phi',\psi) + (d\phi,\psi) + \langle\phi,\psi\rangle,$$
$$(\phi,\psi) = \int_I \phi\psi dx, \tag{1.6}$$
$$\langle\phi,\psi\rangle = \sigma_0\phi(0)\psi(0) + \sigma_1\phi(1)\psi(1).$$

The adjoint problem of Problem (1.1) is the following:

$$L^*w = -(a(x)w')' - (b(x)w)' + d(x)w = g,$$
$$a(0)w'(0) - (b(0) + \sigma_0)w(0) = 0, \tag{1.7}$$
$$a(1)w'(1) + (b(1) + \sigma_1)w(1) = 0.$$

Problem (1.7) can be posed as

$$B^*(w,v) = (g,v), \quad \forall v \in H^1(I) \tag{1.8}$$

where

$$B^*(\phi,\psi) = B(\psi,\phi). \tag{1.9}$$

From the theory of O.D.E.'s and Green's function expression of the solution of the boundary-value problem ([1]), we can assert that there exist C and C^* such that the following hold:

1. For each $f \in L^2(I)$, the solution, $u(x)$, of Problem (1.1) satisfies (1.5) and

$$\|u\|_{H^2(I)} \leq C\|f\|_{L^2(I)}.$$

2. For each $g \in L^2(I)$, Problem (1.7) and thus Problem (1.8) have a unique solution W and

$$\|W\|_{H^2(I)} \leq C^*|g|_{L^2(I)}.$$