PERTURBATION BOUNDS FOR THE POLAR FACTORS*[1)]

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Abstract

Let $A, \tilde{A} \in \mathbb{C}^{m\times n}, \mathrm{rank}(A) = \mathrm{rank}(\tilde{A}) = n$. Suppose that $A = QH$ and $\tilde{A} = \tilde{Q}\tilde{H}$ are the polar decompositions of A and $\tilde{A}$, respectively. It is proved that

$$\|\tilde{Q} - Q\|_F \le 2\|A^\dagger\|_2\|\tilde{A} - A\|_F$$

and

$$\|\tilde{H} - H\|_F \le \sqrt{2}\,\|\tilde{A} - A\|_F$$

hold, where $A^\dagger$ is the Moore-Penrose inverse of A, and $\| \ \|_2$ and $\| \ \|_F$ denote the spectral norm and the Frobenius norm, respectively.

§1. Introduction

In this paper, we use the following notation. The symbol $\mathbb{C}^{m\times n}$ denotes the set of complex $m \times n$ matrices, and $\mathbb{R}^{m\times n}$ the set of real $m \times n$ matrices. A^T and A^H stand for the transpose and the conjugate transpose of A, respectively. $A^\dagger$ is the Moore-Penrose inverse of A. $I^{(n)}$ is the identity matrix of order n. $\| \ \|_2$ denotes the spectral norm and $\| \ \|_F$ the Frobenius norm.

The polar decomposition has found many important applications in factor analysis, aerospace computations and optimization. The following polar decomposition theorem is well known.

Theorem 1.1. *Let $A \in \mathbb{C}^{m\times n}, m \ge n$. Then there exists a matrix $Q \in \mathbb{C}^{m\times n}$ and a unique Hermitian positive semi-definite matrix $H \in \mathbb{C}^{n\times n}$ such that*

$$A = QH, \qquad Q^H Q = I^{(n)}. \tag{1.1}$$

If $\mathrm{rank}(A) = n$, then H is positive definite and Q is uniquely determined.

Let $A \in \mathbb{C}^{m\times n}, m \ge n$, have the singular value decomposition

$$A = U\begin{pmatrix}\Sigma \\ 0\end{pmatrix}V^H$$

where $U = (U_1, U_2) \in \mathbb{C}^{m\times m}, V \in \mathbb{C}^{n\times n}$ are unitary, and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$, $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n$. Then $A = QH$ is the polar decomposition of A, where

$$Q = U_1V^H, \quad H = V\Sigma V^H. \tag{1.2}$$

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In the practical computation, because of the restriction of finite decision, the computed polar factors are those of a matrix $\tilde{A}$ perturbed from A. So it is of interest both for theoretical and for practical purposes to determine the perturbation bounds for the polar factors of a matrix. Higham [1] and Mao [2] have studied that question, and the following results were given.

Theorem 1.2[1]. *Let $A \in \mathbb{C}^{n\times n}$ be nonsingular, with the polar decomposition $A = QH$. If $\varepsilon = \frac{\|\Delta A\|_F}{\|A\|_F}$ satisfies $\kappa_F(A)\varepsilon < 1$, then $A + \Delta A$ has the polar decomposition*

$$A + \Delta A = (Q + \Delta Q)(H + \Delta H),$$

where

$$\frac{\|\Delta H\|_F}{\|H\|_F} \le \sqrt{2}\varepsilon + O(\varepsilon^2), \tag{1.3}$$

$$\frac{\|\Delta Q\|_F}{\|Q\|_F} \le (1+\sqrt{2})\kappa_F(A)\varepsilon + O(\varepsilon^2), \tag{1.4}$$

$$\kappa_F(A) = \|A\|_F\|A^\dagger\|_F.$$

Theorem 1.3.[2]. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular, which has singular value decomposition $A = U\Sigma V^T$, where A is perturbed to $\tilde{A}$, which has singular value decomposition $\tilde{A} = \tilde{U}\tilde{\Sigma}\tilde{V}^T$. Then*

$$\|\tilde{U}\tilde{V}^T - UV^T\|_F \le 2\|A^\dagger\|_2\|\tilde{A} - A\|_F. \tag{1.5}$$

This paper will further study the perturbation bounds for polar factors.

§2. Main Results

First, we introduce the following lemmas:

Lemma 2.1. *Let $B \in \mathbb{C}^{m\times m}, C \in \mathbb{C}^{n\times n}, m \ge n$, be normal matrices, and*

$$\Gamma = \begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & 0 & & \gamma_n \end{pmatrix} \in \mathbb{C}^{m\times n}, \gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_n \ge 0.$$

Then

$$\|B\Gamma - \Gamma C\|_F \ge \gamma_n \left\| B\begin{pmatrix} I^{(n)} \\ 0 \end{pmatrix} - \begin{pmatrix} I^{(n)} \\ 0 \end{pmatrix} C \right\|_F. \tag{2.1}$$

Proof. Let

$$\hat{\Gamma} = \begin{pmatrix} \gamma_1 & & & & O \\ & \gamma_2 & & & \\ & & \ddots & & \\ & O & & \gamma_n & \\ & & & & \gamma_n I^{(m-n)} \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix},$$

where $N \in \mathbb{C}^{(m-n)\times(m-n)}$ is any normal matrix. Then we have

$$\begin{aligned} \|B\hat{\Gamma} - \hat{\Gamma}\hat{C}\|_F^2 &= \left\| B\left(\Gamma, \begin{pmatrix} 0 \\ \gamma_n I^{(m-n)} \end{pmatrix}\right) - \left(\Gamma, \begin{pmatrix} 0 \\ \gamma_n I^{(m-n)} \end{pmatrix}\right)\begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix} \right\|_F^2 \\ &= \left\| \left(B\Gamma - \Gamma C, B\begin{pmatrix} 0 \\ \gamma_n I^{(m-n)} \end{pmatrix} - \begin{pmatrix} 0 \\ \gamma_n I^{(m-n)} \end{pmatrix} N\right) \right\|_F^2 \\ &= \left\| B\Gamma - \Gamma C \right\|_F^2 + \gamma_n^2 \left\| B\begin{pmatrix} 0 \\ I^{(m-n)} \end{pmatrix} - \begin{pmatrix} 0 \\ N \end{pmatrix} \right\|_F^2 \end{aligned} \tag{2.2}$$