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THE DRAZIN INVERSE OF HESSENBERG MATRICES*

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Abstract

The Drazin inverse of a lower Hessenberg matrix is considered. If A is a singular lower Hessenberg matrix, and $a_{i,i+1} \neq 0, i = 1, 2, \cdots, n-1$, then A^D can be given, and expressed explicitly by elements of A. The structure of the Drazin inverse of a lower Hessenberg matrix is also studied.

§1. Introduction

The Drazin inverse A^D of A can be characterized as the unique matrix satisfying the three equations

$$XAX = X, \tag{1.1a}$$

$$AX = XA, \tag{1.1b}$$

$$A^{k+1}X = A^k, \ \text{Index}(A) = k. \tag{1.1c}$$

If $k = 1$, then the Drazin inverse of A is called the group inverse of A and denoted by $A^{\#}$.

The Drazin inverse has been shown to have numerous applications[1]. In [2], it is used to give a closed form for solutions of systems of linear differential equations with singular coefficient matrices. In [5], $A^{\#}$ is used to study finite Markov chains. See [1] for an extensive discussion of the Drazin inverse.

Since an arbitrary square matrix can be reduced to Hessenberg form by means of unitary similarity transformations, and the Drazin inverse is well behaved with respect to similarity, that is $(P^{-1}AP)^D = P^{-1}A^DP$, the study of the Drazin inverse of a Hessenberg matrix is important.

Yasuhiko Ikebe[4] studied the structure of the inverse of a Hessenberg matrix. Cao Wei-lu and W. J. Stewart[3] generalized some results of [4]. Miao Jian-ming[6] presented the form of the group inverse of a Hessenberg matrix, and also generalized the results to more general matrices, called block upper (lower) s-diagonal matrices.

In this paper, we shall present the form of the Drazin inverse of a singular lower Hessenberg matrix with $a_{i,i+1} \neq 0, i = 1, 2, \cdots, n-1$.

§2. Notations and Preliminary Results

Throughout this paper, let $A = (a_{ij})_1^n$ be a lower Hessenberg matrix of order n, and $a_{i,i+1} \neq 0, i = 1, 2, \cdots, n-1$. Let A be partitioned as

$$A = \left(\begin{array}{c:cccc} a_{11} & a_{12} & & & \\ \vdots & \vdots & \ddots & & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ \hdashline a_{n1} & a_{n2} & \cdots & a_{n,n} \end{array}\right) \equiv \begin{pmatrix} c & B \\ \alpha & d^* \end{pmatrix}. \tag{2.1}$$

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Clearly B is a lower triangular matrix, and B^{-1} exists. Let

$$T = \begin{pmatrix} 0 & 0 \\ B^{-1} & 0 \end{pmatrix}, \tag{2.2}$$

$$x = \begin{pmatrix} 1 \\ -B^{-1}c \end{pmatrix}, \tag{2.3}$$

$$y^* = (-d^* B^{-1}, 1). \tag{2.4}$$

Then we have

Lemma 1[3]. *If $\alpha_1 = d^* B^{-1} c - \alpha \neq 0$, then A is nonsingular, and*

$$A^{-1} = T - \alpha_1^{-1} x y^*. \tag{2.5}$$

If $\alpha_1 = 0$, then A is singular.

Lemma 2. *Let*

$$\alpha_1 = d^* B^{-1} c - \alpha, \quad \alpha_i = y^* T^{i-2} x, \quad i = 2, 3, \cdots. \tag{2.6}$$

If $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0, \alpha_{k+1} \neq 0$, then $\text{Index}(A) = k$.

Proof. By [1, p.138], $\text{Index}(A) = k$ is equivalent to saying k is the smallest nonnegative integer such that the limit

$$\lim_{\lambda \to 0} \lambda^k (A + \lambda I)^{-1} \tag{2.7}$$

exists. Now using Lemma 1 gives

$$(A + \lambda I)^{-1} = T(\lambda) - \alpha_1^{-1}(\lambda) x(\lambda) y^*(\lambda). \tag{2.8}$$

where $\lim_{\lambda \to 0} T(\lambda) = T, \lim_{\lambda \to 0} x(\lambda) = x, \lim_{\lambda \to 0} y^*(\lambda) = y^*$. Hence k is the smallest nonnegative integer such that the limit

$$\lim_{\lambda \to 0} \lambda^k \alpha_1^{-1}(\lambda) \tag{2.9}$$

exists. Let $J_0 \in C^{(n-1)\times(n-1)}$, and $f_1, f_{n-1} \in C^{(n-1)\times 1}$ denote the matrices

$$J_0 = \begin{pmatrix} 0 & & 0 \\ 1 & 0 & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}, \quad f_1 = (1, 0, \cdots, 0)^T, f_{n-1} = (0, \cdots, 0, 1)^T.$$

Then

$$\alpha_1(\lambda) = (d^* + \lambda f_{n-1}^T)(B + \lambda J_0)^{-1}(c + \lambda f_1) - \alpha. \tag{2.10}$$

Let $N = B^{-1} J_0$. Then N is nilpotent of index $n - 1$. Hence

$$(B + \lambda J_0)^{-1} = (I + \lambda N)^{-1} B^{-1} = [I - \lambda N + \lambda^2 N^2 - \cdots + (-1)^{n-2} \lambda^{n-2} N^{n-2}] B^{-1}. \tag{2.11}$$

By use of (2.11), (2.10) becomes

$$\alpha_1(\lambda) = \alpha_1 - \beta_2 \lambda + \beta_3 \lambda^2 - \cdots + (-1)^n \beta_{n+1} \lambda^n, \tag{2.12}$$

where

$$\beta_2 = (d^* B^{-1} J_0 - f_{n-1}^T) B^{-1} c - d^* B^{-1} f_1, \tag{2.13a}$$

$$\beta_i = (d^* B^{-1} J_0 - f_{n-1}^T) N^{i-3} B^{-1} (J_0 B^{-1} c - f_1), \quad i = 3, 4, \cdots, n+1. \tag{2.13b}$$