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EXISTENCE AND UNIQUENESS OF MATRIX PADE APPROXIMANTS*1)

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Abstract

For the problems of the left and right matrix Padé approximations, we give the necessary and sufficient conditions for the existence of their solutions. If the left Padé approximant exists, then we prove that its uniqueness is equivalent to the existence of right Padé approximants, and we further give the exact result about the dimension of the linear space ${}^{L}R^{*}(m,n)$ formed from the left Padé approximants.

§1. Introduction

Let

$$f(z)=\sum_{i=0}^{\infty}c_i z^i,\quad c_i\in \mathbf{C}^{d\times d},$$

$$H_k=\{\sum_{i=0}^{k}a_i z^i : a_i\in \mathbf{C}^{d\times d}\},$$

where $\mathbf{C}^{d\times d}$ consists of all $d\times d$ complex matrices with $d>0$. We define right Padé approximants ${}^{R}[m/n]_f={}^{R}P\,{}^{R}Q^{-1}$ by

$$f(z)\,{}^{R}Q(z)-{}^{R}P(z)=O(z^{m+n+1}),\qquad {}^{R}Q(0)=I \tag{1.1}$$

and left-handed Padé approximants ${}^{L}[m/n]_f={}^{L}Q^{-1}\,{}^{L}P$ by

$$ {}^{L}Q(z)f(z)-{}^{L}P(z)=O(z^{m+n+1}),\qquad {}^{L}Q(0)=I \tag{1.2}$$

where $({}^{R}P,\ {}^{R}Q),({}^{L}P,\ {}^{L}Q)\in H_m\times H_n$, and $I\in \mathbf{C}^{d\times d}$ is a unit matrix.

The approach to matrix Padé approximants adopted here follows that of Bessis [1]. For other approaches and generalizations to a non-commutative algebra, we refer to [1] and [2]. For their applications in many domains such as the theoretical physics, the realization problem in system theory, and many other problems such as algebraic properties, computations and convergence of matrix Padé approximants, we refer to the references of [3]. However, the most basic problems, i.e., the existence and uniqueness for matrix Padé approximants, have not yet been investigated completely.

In this paper, the questions concerning the existence, and uniqueness, or nonuniqueness, for matrix Padé approximants are discussed. Some interesting results are established by careful analysis.

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§2. Existence

We shall first quote the following notations:

$$H(i,j,k)=\begin{bmatrix} c_i & c_{i-1} & \cdots & c_{i-j} \\ c_{i+1} & c_i & \cdots & c_{i-j+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{i+k} & c_{i+k-1} & \cdots & c_{i+k-j} \end{bmatrix}, H_T(i,j,k)=\begin{bmatrix} c_i^T & c_{i-1}^T & \cdots & c_{i-j}^T \\ c_{i+1}^T & c_i^T & \cdots & c_{i-j+1}^T \\ \cdots & \cdots & \cdots & \cdots \\ c_{i+k}^T & c_{i+k-1}^T & \cdots & c_{i+k-j}^T \end{bmatrix},$$

where c_t^T is the transpose of c_t , the coefficients of $f(z)$, and define $c_t = 0$, if $t < 0$.

Lemma 2.1. *If $c_t \in \mathbf{C}^{d\times d}$, then*

$$\text{rank } H(i,j,k) = \text{rank } H_T(i+k-j,k,j),$$

where rank denotes the rank of a matrix.

Proof. Since

$$\begin{bmatrix} & & I \\ O & I & \\ I & O & \end{bmatrix} H^T(i,j,k) \begin{bmatrix} & & I \\ O & I & \\ I & O & \end{bmatrix} = H_T(i+k-j,k,j),$$

by the relation $\text{rank } H(i,j,k) = \text{rank } H^T(i,j,k)$, the lemma is valid.

Now we establish the existence results.

Theorem 2.1. *Let $f(z) = \sum_{i=0}^{\infty} c_i z^i$, $c_i \in \mathbf{C}^{d\times d}$. Then*

(i) $^R[m/n]_f$ *exists if and only if*

$$\text{rank } H(m,n-1,n-1) = \text{rank } H(m+1,n,n-1). \tag{2.1}$$

(ii) $^L[m/n]_f$ *exists if and only if*

$$\text{rank } H(m,n-1,n-1) = \text{rank } H(m,n-1,n). \tag{2.2}$$

(iii) *Both $^R[m/n]_f$ and $^L[m/n]_f$ exist, if $H(m,n-1,n-1)$ is nonsingular* (see [5]).

Proof. (i) Let $^RP(z) = \sum_{i=0}^{m} {}^Ra_i z^i$, $^RQ(z) = \sum_{i=0}^{n} {}^Rb_i z^i$, ^Ra_i, $^Rb_i \in \mathbf{C}^{d\times d}$. Then by equating the coefficients of z^i in (1.1) for $i = 0, 1, \ldots, m+n$, one has

$$\begin{bmatrix} {}^Ra_0 \\ {}^Ra_1 \\ \vdots \\ {}^Ra_m \end{bmatrix} = H(0,n,m) \begin{bmatrix} {}^Rb_0 \\ {}^Rb_1 \\ \vdots \\ {}^Rb_n \end{bmatrix} \tag{2.3}$$

and

$$H(m,n-1,n-1) \begin{bmatrix} {}^Rb_1 \\ {}^Rb_2 \\ \vdots \\ {}^Rb_n \end{bmatrix} = \begin{bmatrix} -c_{m+1} \\ -c_{m+2} \\ \vdots \\ -c_{m+n} \end{bmatrix}. \tag{2.4}$$