Journal of Computational Mathematics, Vol.8, No.2, 1990, 108–117.

COMPUTATION OF MINIMAL AND QUASI–MINIMAL SUPPORTED BIVARIATE SPLINES *

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When multivariate splines are needed to approximate the solution of a certain problem, to generate surfaces or solids, to analyze discrete data, etc., with specified grid and degree of smoothness, it is important to seek locally supported (ls) elements whose polynomial pieces have the lowest total degree. In addition, it is usually desirable to select those with minimal supports (ms). In [1] and [2], de Boor and Höllig studied ms splines in the 3-and 4-direction meshes and pointed out that their study of the 4-direction mesh is not complete. In [3], we have shown that even in the 4-direction mesh, it may happen that there are too few ms bivariate splines to generate all the ls ones. Hence, the notion of quasi-minimal supported (qms) splines was introduced. In this paper, the uniqueness problem is discussed, and recurrence relations as well as computational schemes for both the 3-direction and 4-direction meshes will be given. One interesting property of ms and qms splines is that all of them in the same space are needed to form a partition of unity. We will also characterize those spaces which are spanned by these ls functions.

As usual, $\Delta^{(1)}$ and $\Delta^{(2)}$ will denote the 3-and 4-direction meshes in R^2 with integral grid points, respectively, and $S^k_m(\Delta^{(i)})$, $i=1,2$, the spaces of functions in C^k whose restricions on the triangular cells are polynomials of degree m.

The number of "independent" locally supported functions in $S^k_m(\Delta^{(i)})$ will be called the locally supported spline cardinality of $S^k_m(\Delta^{(i)})$, denoted by # lss of $S^k_m(\Delta^{(i)})$, $i=1,2$. It is well known (cf. [6]) that

$$\#\ \text{lss of } S^k_m(\Delta^{(1)}) = d^k_m(3) = (m-k-[\frac{k+1}{2}])_+(m-2k+[\frac{k+1}{2}]) \qquad (1)$$

and

$$\#\ \text{lss of } S^k_m(\Delta^{(2)}) = d^k_m(4) = \frac{1}{2}(m-k-[\frac{k+1}{3}])_+(3m-5k+1+[\frac{k+1}{3}]) \qquad (2)$$

where $[x]$ denotes, as usual, the integer part of x.

In order that $S^k_m(\Delta^{(i)})$ may be useful for approximation purposes, we must have positive #lss. In addition, given the smoothness condition C^k, the lowest degree m is desirable. Following de Boor and Höllig [2], we set

$$d_i(k) = \text{the smallest } \ m \ \text{ such that } d^k_m(i+2) > 0.$$

From (1) and (2), we obtain

*Received June 26, 1987.

k	2r-1	2r
$d_1(k)$	$3r$	$3r+1$
#lss	2	1

3-direction mesh $\Delta^{(1)}$
r=0,1, $\cdots$

k	$3r$	$3r+1$	$3r+2$
$d_2(k)$	$4r+1$	$4r+2$	$4r+4$
#lss	2	1	3

4-direction mesh $\Delta^{(2)}$
r=0,1, $\cdots$

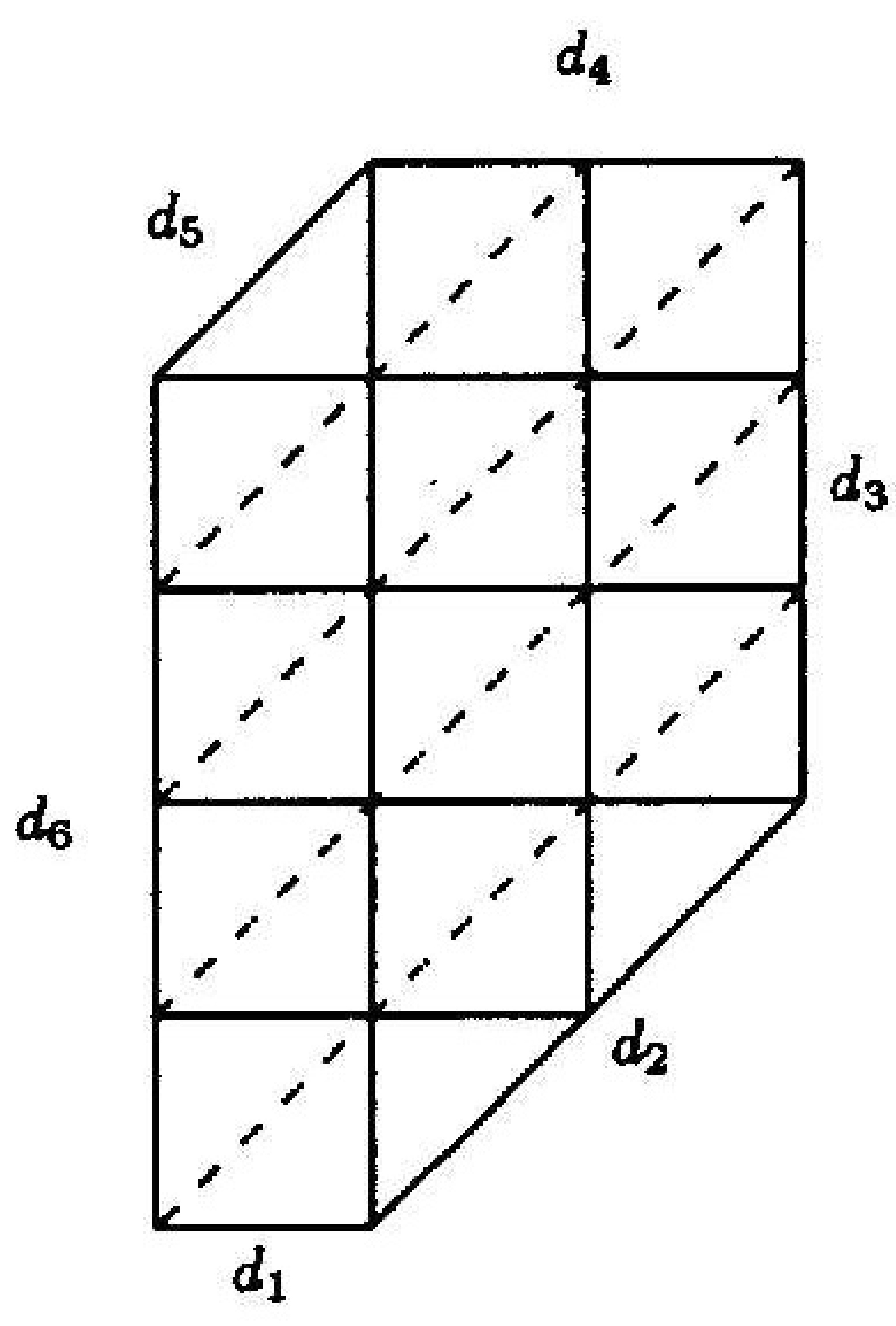
If g is an ls function in $S^k_{d_1(k)}(\Delta^{(1)})$ whose support is a convex polygon, we will denote its support by

$$\text{supp } g = \{d_1, \cdots, d_6\}$$

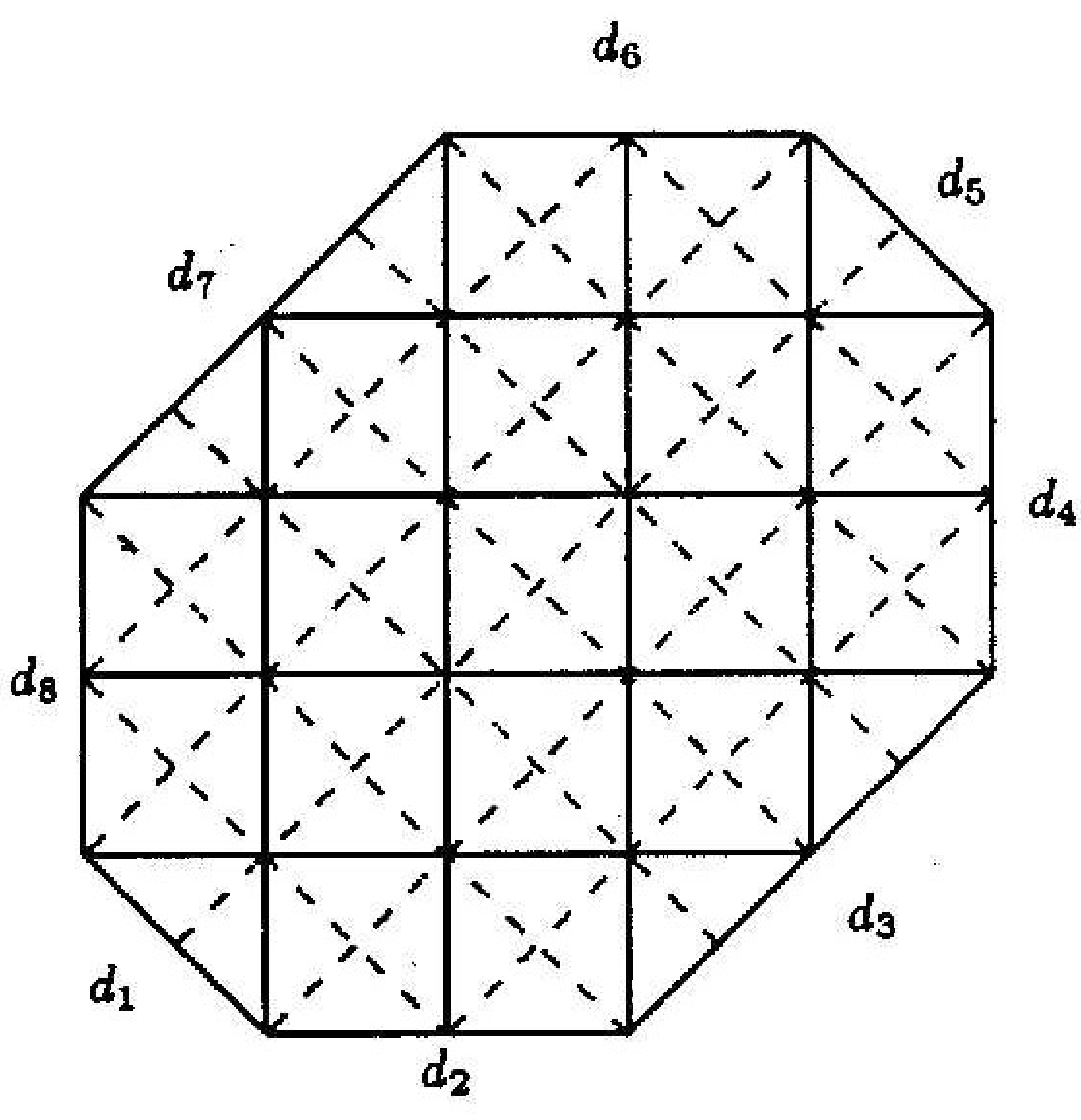
where $d_1, \cdots, d_6$ are nonnegative integers, indicating the number of units (i.e. horizontal or vertical edges, or diagonals) of the partition $\Delta^{(1)}$ in the " directions " $e_1, e_3, e_2, -e_1, -e_3, -e_2$, respectively, where $e_1 = (1,0), e_2 = (0,1), e_3 = (1,1)$. If f is an ls function in $S^k_{d_2(k)}(\Delta^{(2)})$ whose support is a convex polygon, it is clear that none of its vertices lie on a grid point determined by only two grid lines, and its support will be denoted by

$$\text{supp} f = \{d_1, \cdots, d_8\}$$

where $d_1, \cdots, d_8$ are nonnegative integers, indicating the number of units of the partition $\Delta^{(2)}$ in the "directions" $e_4, e_1, e_3, e_2, -e_4, -e_1, -e_3, -e_2$, respectively, where $e_4 = (1, -1)$. These are shown in the followig figures.



3-direction mesh



4-direction mesh

Following [4], we denote by g^r_i and f^r_i the ls functions of $S^k_{d_1(k)}(\Delta^{(1)})$ and $S^k_{d_2(k)}(\Delta^{(2)})$, respectively, with minimal or quasi-minimal supports. In addition, we will call g^0_i and f^0_i the initial ones.