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UPPER BOUNDS OF THE SPECTRAL RADII OF SOME ITERATIVE MATRICES*

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Abstract

In this paper, the concept of optimally scaled matrix and the estimate of $\|M^{-1}N\|_\infty$ in our previous paper are used to find the upper bounds of the spectral radii of the iterative matrices SOR, SSOR, AOR and SAOR. The sharpness of the upper bounds of the spectral radii of SOR and AOR is established. The proofs are very intuitive and may be considered as the geometrical interpretations of our theorems.

§1. Introduction

It is well-known that if the coefficient matrix A of a system of linear algebraic equations

$$Ax = f \tag{1}$$

is a nonsingular H-matrix and

$$0 \le \omega \le 2/[1 + S(|J|)], \tag{2}$$

then the spectral radius $S(L_\omega^A)$ of the SOR iterative matrix

$$L_\omega^A = (D - \omega L)^{-1}[(1-\omega)D + \omega U] \tag{3}$$

satisfies (see [1], for example)

$$S(L_\omega^A) \le |1-\omega| + \omega S(|J|) =: \delta \tag{4}$$

where $D = \mathrm{diag}(A), B = D - A, L$ and U are lower and upper triangular matrices of B respectively and

$$J = D^{-1}B \tag{5}$$

is the Jacobian iterative matrix of A.

In [2], it is proved that the upper bound in (4) is sharp, that is, given $v \in [0,1)$ and $\omega \in [0, 2/(1+v)]$, the equality

$$\sup_{A \in H_v} \{S(L_\omega^A)\} = |1-\omega| + \omega v$$

holds, where H_v is the set of all nonsingular H-matrices with

$$v = S(|J|), \tag{6}$$

which is obviously less than one, and L_ω^A is the SOR iterative matrix of A.

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For the symmetric SOR (SSOR) iterative matrix

$$S_\omega^A = U_\omega^A L_\omega^A, \tag{7}$$

where

$$U_\omega^A = (D - \omega U)^{-1}[(1-\omega)D + \omega L], \tag{8}$$

we have from [3]

$$S(S_\omega^A) \le |1-\omega| + \omega S(|J|). \tag{9}$$

In this paper, we use the theorems abour $\|M^{-1}N\|_\infty$ and the optimally scaled matrix in [4] to derive the upper bound of L_ω^A and S_ω^A and then generalize our results to the AOR and SAOR matrices

$$L_{\tau,\omega}^A = (D - \tau L)^{-1}[(1-\omega)D + (\omega-\tau)L + \omega U], \tag{10}$$

and

$$S_{\tau,\omega}^A = U_{\tau,\omega}^A L_{\tau,\omega}^A, \tag{11}$$

where

$$U_{\tau,\omega}^A = (D - \tau U)^{-1}[(1-\omega)D + (\omega-\tau)U + \omega L]. \tag{12}$$

We obtain

$$S(S_\omega^A) \le (|1-\omega| + \omega v)^2, \tag{13}$$

$$S(L_{\tau,\omega}^A) \le |1-\omega| + \omega v \tag{14}$$

and

$$S(S_{\tau,\omega}^A) \le (|1-\omega| + \omega v)^2. \tag{15}$$

The upper bound in (13) is obviously better than in (9).

Further, we prove also the sharpness of the upper bounds of $S(L_\omega^A)$ and $S(L_{\tau,\omega}^A)$. Our method is very intuitive and may be considered as a geometrical interpretation of the upper bounds and their sharpness. Moreover, the matrices used here are more general than those in [3].

§2. The Upper Bounds of the Spectral Radii of $L_\omega^A, S_\omega^A, L_{\tau,\omega}^A$ and $S_{\tau,\omega}^A$

First, for completeness, we present the theorems in [4], which will be used here, as our lemmas:

Lemma 1. *If $M = (m_{ij})$ and $N = (n_{ij})$ are $n \times n$ matrices and*

$$|m_{ii}| > \sum_{j \ne i} |m_{ij}|, \quad i = 1, 2, \cdots, n, \tag{16}$$

then

$$\|M^{-1}N\|_\infty \le \max_i \Big[\sum_j |n_{ij}| / (|m_{ii}| - \sum_{j \ne i} |m_{ij}|\Big]. \tag{17}$$

Lemma 2. *If A is an irreducible matrix, $D = \mathrm{diag}(A)$ is nonsingular and $B = D - A$, then there is a positive diagonal matrix $Q = \mathrm{diag}(q_1, q_2, \cdots, q_n)$ such that the matrix*

$$\tilde{A} = (\tilde{a}_{ij}) = AQ \tag{18}$$