Journal of Computational Mathematics, Vol.8, No.3, 1990, 261-275.

PROJECTIVE APPROXIMATION OF DOUBLE LIMIT POINTS FOR NONLINEAR PROBLEMS *[1)]

Ma Ya-nan
(*Wuhan University, Wuchang, China*)

Abstract

In [2], general approximation results for the solutions in a neighborhood of a simple limit point are given. In this paper we give projective approximation results for the solutions in a neighborhood of a double limit point. Application of these results to a nonlinear partial differential equation and numerical results are given.

§1. Introduction

Consider a nonlinear problem of the form

$$F(\lambda, u) = 0 \tag{1.1}$$

where $F : R \times V \to V$ is sufficiently smooth, and V is a Hilbert space. In [2], finite dimensional approximation of branches of solutions of problem (1.1) in a neighborhood of a simple limit point and a simple bifurcation point have been studied. In this paper, we will discuss the projective approximation of branches of solutions of problem (1.1) in a neighborhood of a double limit point (λ_0, u_0) of F, i.e., a point $(\lambda_0, u_0) \in R \times V$ which satisfies the following properties:

1) $F(\lambda_0, u_0) = 0$;
2) $D_uF(\lambda_0, u_0)$ is singular and dim Ker $D_uF(\lambda_0, u_0) =$ codim Range $D_uF(\lambda_0, u_0) = 2$;
3) $D_\lambda F(\lambda_0, u_0) \notin$ Range $D_uF(\lambda_0, u_0)$.

An outline of the paper is as follows. In Section 2, we give a local analysis of a double limit point. In Section 3 we consider the projective approximation problem of (1.1) near the double limit point. Using the method similar to that in [2], we obtain the error estimates and convergence results of the solution sets. In Section 4, we apply our results to a simple example, and give numerical results.

§2. Local Analysis of Double Limit Points

Consider the nonlinear problem

$$F(\lambda, u) \equiv u + TG(\lambda, u) = 0 \tag{2.1}$$

where $T \in \mathcal{L}(V, V)$, and $G \in C^r (r \geq 3) : R \times V \to V; V$ is a Hilbert space.

We assume that $(\lambda_0, u_0) \in R \times V$ is a double limit point of F in the sense that

1) $F^0 \equiv F(\lambda_0, u_0) = 0$; (2.2)

* Received November 27, 1987.
1) The Project Supported by National Natural Science Foundation of China.

2) $D_uF^0 \equiv D_uF(\lambda_0, u_0) = I + TD_uG^0 \in \mathcal{L}(V,V), -1$ is an eigenvalue of TD_uG^0 with algebraic multiplicity 2;

3) $D_\lambda F^0 \equiv D_\lambda F(\lambda_0, u_0) \notin$ Range (D_uF^0).

Moreover, we assume that

$$\text{Range } (D_uF^0) \text{ is closed}; \quad D_uF^0 \text{ is self-adjoint.} \tag{2.3}$$

Remark. Under the assumptions that T is compact and F is symmetric in some sense, Raugel [5] has discussed multiple limit point problems. Here discarding the above assumptions, we only assume that (2.3) holds. We notice that condition (2.3) holds if D_uF^0 is a Fredholm operator and self-adjoint. Particularly, (2.3) holds for T compact and D_uF^0 self-adjoint.

From 2) of (2.2) and the properties of self-adjoint operators, it follows that

$$\text{Ker } (D_uF^0) = \text{ Ker } ((D_uF^0)^n), \quad n = 2, 3, \cdots.$$

Hence we can find $\varphi_1, \varphi_2 \in V, (\varphi_i, \varphi_j) = \delta_{ij}, i, j = 1, 2$, such that

$$\text{Ker}(D_uF^0) = \text{span}\{\varphi_1, \varphi_2\}.$$

By the closed range theorem[1], we have

$$\text{Range } (D_uF^0) = \text{Ker}(D_uF^0) = \{v \in V : (v, \varphi_i) = 0, \quad i = 1, 2\}.$$

Set

$$V_1 = \text{ Ker } (D_uF^0), \quad V_2 = \text{ Range } (D_uF^0).$$

Then $V = V_1 + V_2$, and D_uF^0 is an isomorphism of V_2.

From 3) of (2.2), without loss of generality, we assume

$$(D_\lambda F^0, \varphi_1) = (TD_\lambda G^0, \varphi_1) \neq 0.$$

Now we define the projective operator $Q : V \to V_2$ by

$$Qv = v - \sum_{i=1}^{2}(v, \varphi_i)\varphi_i, \quad v \in V.$$

Then equation (2.1) is equivalent to the system

$$\begin{cases} QF(\lambda, u) = 0, \\ (I - Q)F(\lambda, u) = 0. \end{cases} \tag{2.4}$$

Given $u \in V$, there exists a unique decomposition of the form

$$u = u_0 + \sum_{i=1}^{2} \xi_i\varphi_i + v, \quad \xi_i \in R, \quad i = 1, 2, \quad v \in V_2.$$

Setting $\xi = (\xi_1, \xi_2)$, the first equation of (2.4) becomes

$$\mathcal{F}(\lambda, \xi, v) \equiv QF(\lambda, u_0 + \sum_{i=1}^{2} \xi_i\varphi_i + v) = 0. \tag{2.5}$$