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ON AN UNCONDITIONALLY STABLE SCHEME FOR THE UNSTEADY NAVIER–STOKES EQUATIONS*

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Abstract

Theoretical time step constraints of semi-implicit schemes are known to be more restrictive than should be in practice. We intend to alleviate the constraints with more smoothness assumptions on the solutions. By introducing a new scheme with modification on the treatment of the nonlinear term, we are able to prove that the scheme is unconditionally stable and convergent. Further more, we show that the modified scheme and the original semi-implicit one are equivalent under a weak condition on the time step and the number of space discretization points.

§1. Introduction

In numerical simulations of incompressible flow represented by the Navier-Stokes equations (1.1), one of the major difficulties is to construct a suitable time discretization scheme. The origin of such difficulty consists essentially of two parts:

(i) The pressure and the velocity in Navier-Stokes equations are coupled by the incompressibility constraint (1.1b) such that a direct inversion of the resulting discrete system is very expensive. A great number of fast Stokes solvers have been developed by using either an iterative method or a Green's function method (also called influence matrix method, see for instance [8]). Another remedy for removing this difficulty is to use the so called projection method initially proposed by A.J.Chorin and R.Temam (cf. [4], [11]) which separates the calculation of the pressure from that of the velocity. However, this kind of splitting schemes suffers from a large time splitting error which can only be removed by a sophisticated extrapolation process (cf. [10]).

(ii) The treatment of the nonlinear term: usually, explicit treatment of the nonlinear term leads to in some cases a restrictive theoretical time step constraint (see for instance [12]) while implicit treatment makes the resulting discrete system very difficult to be resolved.

In this paper, we concentrate on improving existing theoretical stability constraints for semi-implicit schemes in which the diffusion term is treated implicitly, leaving the convection term (i.e. nonlinear term) treated explicitly.

In many cases, one observes that a semi-implicit scheme gives stable results under a time step constraint which is much weaker than what the theoretical results predict, especially in cases where a smooth solution exists. A natural question one can ask is: can we improve the existing stability conditions by giving more smoothness assumptions on the solutions?

We will give a positive answer to this question by considering a concrete space discretization, namely, the Chebyshev-Galerkin approximation (we refer to [7] for a detailed presentation of this method). For other space discretizations, similar results could be obtained by

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using the same technique. The idea used here can also be applied to other time-dependent elliptic nonlinear systems.

The unsteady Navier-Stokes equations in the primitive variable formulation are written as

$$\frac{\partial u}{\partial t} - \nu \Delta u + u \nabla u + \nabla p = f(x,t)\ , \quad (x,t) \in Q = \Omega \times [0,T], \tag{1.1a}$$

$$\text{div } u = 0 \ , \quad \text{in } Q, \tag{1.1b}$$

$$u(x,0) = u_0(x), \tag{1.1c}$$

$$u(x,t) = 0|_{\partial\Omega} \quad \forall t \in [0,T], \tag{1.1d}$$

where Ω is an open set in R^d $(d = 2 \ or \ 3)$ with sufficiently smooth boundary, the unknowns are the vector function u (velocity) and the scalar function p (pressure). For the sake of simplicity, we assume that the velocity satisfies the homogeneous boundary condition.

We will restrict ourselves to the two dimensional case. More specifically, we consider $\Omega = (-1,1) \times (-1,1)$. The Chebyshev weight function defined in Ω is

$$\omega(x) = (1 - x_1^2)^{-\frac{1}{2}}(1 - x_2^2)^{-\frac{1}{2}} \quad \text{for} \quad x = (x_1, x_2) \in \Omega.$$

The following functional spaces will be used in the sequel:

$$\mathcal{X} = \mathcal{H}^1_{0,\omega}(\Omega),$$

$$\mathcal{H}_\omega = \{u \in \mathcal{L}^2_\omega(\Omega) : \quad \text{div } u = 0, \quad u \cdot \vec{n} = 0\},$$

$$\mathcal{V}_\omega = \{u \in \mathcal{X} : \quad \text{div } u = 0\},$$

$$\tilde{\mathcal{V}}_\omega = \{u \in \mathcal{X} : \quad \text{div } (u \cdot \omega) = 0\},$$

where ω is the Chebyshev weight function, and $\mathcal{L}^2_\omega(\Omega)$ and $\mathcal{H}^1_{0,\omega}(\Omega)$ are weighted Sobolev spaces. To alleviate notations, we use calligraphic letters to denote vector function spaces, for instance, $\mathcal{L}^2_\omega = (L^2_\omega)^2$.

With the help of these functional spaces, we can reformulate the problem (1.1) as

$$\begin{cases} \text{find} \quad u(t) \in \mathcal{V}_\omega \quad \text{such that} \\ \frac{\partial}{\partial t}t(u,v)_\omega + \nu a_\omega(u,v) + (B(u),v)_\omega = < f, v >_\omega \ , \quad \forall v \in \tilde{\mathcal{V}}_\omega, \\ u(0) = u_0 \end{cases} \tag{1.2}$$

where we have

$$(u,v)_\omega = (u, v\omega) = \int_\Omega uv\omega dx \ , \ a_\omega(u,v) = (\nabla u, \nabla(v \cdot \omega)),$$

$$B(u) = \sum_{i=1}^{d} u_i \frac{\partial u}{\partial x_i} \ \text{and} \ < \cdot, \cdot >_\omega \ \text{the duality relation between} \ \mathcal{X}' \ \text{and} \ \mathcal{X}.$$

Due to the Chebyshev weight function involved here, the formulation (1.2) is not symmetric such that the existence of solutions for (1.2) is not covered by the conventional theory.