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THE NUMERICAL SOLUTION OF SECOND–ORDER WEAKLY SINGULAR VOLTERRA INTEGRO–DIFFERENTIAL EQUATIONS*

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Abstract

In this paper we investigate the attainable order of (global) convergence of collocation approximations in certain polynomial spline spaces for solutions of a class of second-order volterra integro-differential equations with weakly singular kernels. While the use of quasi-uniform meshes leads, due to the nonsmooth nature of these solutions, to convergence of order less than one, regardless of the degree of the approximating spline function, collocation on suitably graded meshes will be shown to yield optimal convergence rates.

§1. Introduction

In this paper we present an analysis of certain numerical methods for solving the second-order Volterra integro-differential equation (VIDE)

$$y''(t) = f(t, y(t)) + \int_0^t (t-s)^{-\alpha} k(t, s, y(s))ds, \quad t \in I := [0, T], \tag{1.1}$$

with initial conditions $y(0) = y_0, y'(0) = z_0$. Here, the given functions $f : I \times I\!R \to I\!R$ and $k : S \times I\!R \to I\!R$ (with $S := \{(t,s) : 0 \le s \le t \le T\}$) denote given smooth functions, and constant α satisfies $0 < \alpha < 1$. In practical applications one very frequently encounters the linear counterpart of (1.1)

$$y''(t) = p(t)y(t) + q(t) + \int_0^t (t-s)^{-\alpha} K(t,s)y(s)ds, \quad t \in I (0 < \alpha < 1). \tag{1.2}$$

In the subsequent analysis we shall, for ease of exposition, usually utilize the linear version of (1.1) to display the principal ideas.

Equations of type (1.1) (in practical applications one occasionally encounters second-order VIDEs whose right-hand sides contain also terms involving y'; see e.g., [8, 9]; we shall consider this general case in a subsequent paper) arise in many areas of physics and engineering. But the literature on the numerical solution of (1.1) or its general case is comparatively small. Very little convergence analysis has been given so far. Moreover, as far as high-order Volterra integro-differential equations are concerned, Aguilar & Brunner [1] have presented a study of collocation techniques for Eq. (1.1) with $\alpha = 0$, and Tang [10]

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for high-order Volterra integro-differential equations without singularity. Prosperetti [8, 9] introduced methods based on piecewise cubic Hermite interpolation for a class of second-order integro-differential equations, where care is taken that on a suitable initial interval the nonsmooth solution is approximated accurately. No convergence analysis has been given to this method.

The numerical methods to be analyzed will be collocation methods in the polynomial spline space,

$$S_{m+1}^{(1)}(Z_N) := \{u : u|_{\sigma_n} =: u_n \in P_{m+1}, 0 \le n \le N-1,$$
$$u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n) \text{ for } t_n \in Z_N \text{ and } j = 0, 1\}, \tag{1.3}$$

associated with a given partition (or : mesh) Π_N of the interval I,

$$\Pi_N : 0 = t_0^{(N)} < t_1^{(N)} < \cdots < t_N^{(N)} = T$$

(the index indicating the dependence of the mesh points on N will, for ease of notation, subsequently be suppressed). Here, P_{m+1} denotes the space of real polynomials of degree not exceeding $m+1$, and we have set $\sigma_0 := [t_0, t_1]$, $\sigma_n := (t_n, t_{n+1}]$ $(1 \le n \le N-1)$; the set $Z_N := \{t_n : 1 \le n \le N-1\}$ (i.e., the interior mesh points) will be referred to as the knots of these polynomial splines. In addition, we define

$$h := \max\{h_n : 0 \le n \le N-1\}, \quad h' := \min\{h_n : 0 \le n \le N-1\}, \tag{1.4}$$

where $h_n := t_{n+1} - t_n$; the quantity h is often called the diameter of the mesh Π_N (note that, according to the above remark on our notation, both h and h' will depend on N).

In order to describe these collocation methods we rewrite (1.1), for $t \in \sigma_n$, in "one-step form",

$$y''(t) = F_n(y;t) + \int_{t_n}^{t} (t-s)^{-\alpha} k(t,s,y(s))ds, \tag{1.5}$$

where

$$F_n(y;t) := f(t, y(t)) + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t-s)^{-\alpha} k(t,s,y(s))ds. \tag{1.6}$$

For given parameters $\{c_j\}$ with $0 \le c_1 < \cdots < c_m \le 1$, we introduce the sets

$$X_n := \{t_{nj} := t_n + c_j h_n ; 1 \le j \le m\}, \quad 0 \le n \le N-1, \tag{1.7}$$

and we define

$$X(N) := \bigcup_{n=0}^{N-1} X_n;$$

the set $X(N)$ will be referred to as the set of collocation points, while $\{c_j\}$ will be called collocation parameters. A numerical approximation to the exact solution y of (1.1) (or (1.2)) is an element of $S_{m+1}^{(1)}(Z_N)$ satisfying the given equation on $X(N)$, i.e., by (1.5), this approximation u is computed recursively from

$$u_n''(t_{nj}) = F_n(u; t_{nj}) + h_n^{1-\alpha} \int_0^{c_j} (c_j - s)^{-\alpha} k(t_{nj}, t_n + sh_n, u_n(t_n + sh_n))ds,$$
$$j = 1, \cdots, m, \tag{1.8}$$