Journal of Computational Mathematics, Vol.8, No.4, 1990, 333–341.

# INCOMPLETE SEMIITERATIVE METHODS FOR SOLVING OPERATOR EQUATIONS IN BANACH SPACE*[1)]

Kuang Jiao-xun
(*Shanghai Normal University, Shanghai, China*)

**Abstract**

There are several methods for solving operator equations in a Banach space. The successive approximation methods require the spectral radius of the iterative operator be less than 1 for convergence.

In this paper, we try to use the incomplete semiiterative methods to solve a linear operator equation in Banach space. Usually the special semiiterative methods are convergent even when the spectral radius of the iterative operator of an operator equation is greater than 1.

## §1. Introduction

Let $X$ be a complex Banach space. The set of all bounded linear operators from $X$ into $X$ is denoted by $B[X]$ which is also a Banach space. We consider the linear operator equation

$$Ax = b, \tag{1.1}$$

where $A \in B[X]$ and $b \in X$ is given. If $A^{-1} \in B[X]$ then the solution $\hat{x} = A^{-1}b$ of equation (1.1) exists uniquely. To study the successive approximation methods and semiitterative methods, we rewrite equation (1.1) in a fixed point form

$$x = Tx + f, \tag{1.2}$$

where $T \in B[X]$ and $f \in X$.

Let $\sigma(T)$ be the spectrum of $T$. Then for equation (1.2) and therefore equation (1.1) there exists an unique solution if and only if $1 \bar{\in} \sigma(T)$. We assume $1 \bar{\in} \sigma(T)$ and apply the successive approximation method to solve the following operator equation

$$x = Tx + f.$$

The iterative sequence

$$x_{m+1} = Tx_m + f = T^{m+1}x_0 + \Big(\sum_{i=0}^{m} T^i\Big)f, \quad m \geq 0, \quad x_0 \in X \tag{1.3}$$

converges for any $x_0 \in X$ if and only if the solution of equation (1.2) has a Neumann expansion

$$\hat{x} = \Big(\sum_{i=0}^{\infty} T^i\Big)f. \tag{1.4}$$

* Received January 22, 1988.
1) The Project Supported by Institute for Computational Mathematics of Kent State University.

But the Neumann series

$$\sum_{i=0}^{\infty} T^i$$

converges if and only if the spectral radius $r\sigma(T)$ of $T$ satisfies

$$r\sigma(T) < 1. \tag{1.5}$$

This is a very strict condition for operator $T$.

We denote by $\mathcal{F}(T)$ the family of all functions which are analytic on some neighbourhood of $\sigma(T)$ (the neighbourhood need not be connected, and can depend on the particular function $f \in \mathcal{F}(T)$). Let $T \in B[X], f \in \mathcal{F}(T)$ and let $V$ be an open subset of $C$ whose boundary $B$ consists of a finite number of rectifiable Jordan curvers. We assume that $B$ is oriented. Suppose $V \supseteq \sigma(T)$ and $V \cup B$ is contained in the analytic domain of $f$. Then the operator $f(T)$ is defined by equation

$$f(T) = \frac{1}{2\pi i} \int_B f(\lambda)(\lambda - T)^{-1} d\lambda. \tag{1.6}$$

**Proposition 1**[9]. Let $T \in B[X]$ and let $f \in \mathcal{F}(T)$. Then

$$f(\sigma(T)) = \sigma(f(T)), \tag{1.7}$$

and hence

$$r\sigma(T)^n = r\sigma(T^n), \quad n = 1, 2, \cdots. \tag{1.8}$$

## §2. Incomplete Semiiterative Methods

Given a linear equation $Ax = b$, where $A \in B[X], b \in X$, we rewrite $Ax = b$ in a fixed point form

$$x = Tx + f, \tag{2.1}$$

where $T \in B[X], f \in X$ and $1 \bar{\in} \sigma(T)$. Corresponding to the successive approximation method

$$x_m = Tx_{m-1} + f, \tag{2.2}$$

if we define the error vector and the residual vector as

$$e_m := \hat{x} - x_m, \quad r_m := f - (I - T)x_m, \tag{2.3}$$

then there holds

$$e_m := T^m e_0, \quad r_m := T^m r_0. \tag{2.3'}$$

Following Varge[4] we define a semiiterative method (SIM) with respect to iterative method (2.2) by

$$y_m := \sum_{i=0}^{m} \pi_{m,i} x_i, \quad m \geq 0, \tag{2.4}$$

where the infinite lower triangular matrix

$$P := \begin{bmatrix} \pi_{00} & & & O \\ \pi_{10} & \pi_{11} & & \\ \pi_{20} & \pi_{21} & \pi_{22} & \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix} \tag{2.5}$$