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AN IMBEDDING METHOD FOR COMPUTING THE GENERALIZED INVERSES*

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Abstract

This paper deals with a system of ordinary differential equations with known conditions associated with a given matrix. By using analytical and computational methods, the generalized inverses of the given matrix can be determined. Among these are the weighted Moore-Penrose inverse, the Moore-Penrose inverse, the Drazin inverse and the group inverse. In particular, a new insight is provided into the finite algorithms for computing the generalized inverse and the inverse.

§1. Introduction

In [1, 2], the imbedding method for nonlinear matrix eigenvalue problems and for computational linear algebra are presented.

In many engineering problems we must find the generalized inverses of a given matrix.

Let $A \in C^{m\times n}$. Throughout this paper, let M and N be positive definite matrices of order m and n respectively. Then, there is a unique matrix $X \in C^{n\times m}$ satisfying

$$AXA = A, \quad XAX = X, \quad (MAX)^* = MAX, \quad (NXA)^* = NXA. \tag{1.1}$$

This X is called the weighted Moore-Penrose inverse of A, and is denoted by $X = A^+_{MN}$. In particular, when $M = I_m, N = I_n$, the matrix X that satisfies (1.1) is called the Moore-Penrose inverse of A, and is denoted by $X = A^+$, i.e., $A^+ = A^+_{I_m I_n}$.

Let $A \in C^{n\times n}$. The smallest nonnegative integer k such that

$$\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1}) \tag{1.2}$$

is called the index of A, and is denoted by $\mathrm{Ind}(A)$.

Let $A \in C^{n\times n}$. With $\mathrm{Ind}(A) = k$ and if $X \in C^{n\times n}$ is such that

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA \tag{1.3}$$

then X is called the Drazin inverse of A, and is denoted by $X = A_d$. In particular, when $\mathrm{Ind}(A) = 1$, the matrix X that satisfies (1.3) is called the group inverse of A, and is denoted by $X = A^{\#}$.

An imbedding method for the Moore-Penrose inverse is given in [3]. In this paper, the imbedding methods for the weighted Moore-Penrose inverse Moore-Penrose inverse, the Drazin inverse and the group inverse are presented, and these methods have a uniform formula.

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First, we show the generalized inverses can be characterized in terms of a limiting process. These expressions involve the inverse of the matrix $B_t(z)$, where $B_t(z)$ is a matrix of z. Secondly, we show how this problem may be reduced to integrating a system of ordinary differential equations subject to initial conditions. In particular, a new insight is provided into a series of finite algorithms for computing the generalized inverses and the inverse in [4–6, 9].

§2. Generalized Inverses as a Limit

In this section, we will show how the generalized inverses A^+, A^+_{MN}, A_d and $A^\#$ can be characterized in terms of a limiting process respectively.

Theorem 2.1. *Let* $A \in C^{m\times n}$, $\operatorname{rank} A = r$. *Then*

$$A^+_{MN} = \lim_{z\to 0}(N^{-1}A^*MA - zI)^{-1}N^{-1}A^*M \tag{2.1}$$

where z *tends to zero through negative values.*

Proof. From the (M,N)-singular value decomposition theorem[7], there exists an M-unitary matrix $U \in C^{m\times m}$ and an N^{-1}-unitary matrix $V \in C^{n\times n}$ such that

$$A = U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}V^* \tag{2.2}$$

where

$$U^*MU = I_m, \quad V^*N^{-1}V = I_n, \tag{2.3}$$

$$D = \operatorname{diag}(d_1, d_2, \cdots, d_r), \quad d_i > 0, \quad i = 1, 2, \cdots, r \tag{2.4}$$

and

$$A^+_{MN} = N^{-1}V\begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*M. \tag{2.5}$$

Let

$$N^{-1/2}V = \tilde{V} = (v_1, v_2, \cdots, v_n), \tag{2.6}$$

$$M^{1/2}U = \tilde{U} = (u_1, u_2, \cdots, u_m), \tag{2.7}$$

then

$$\tilde{V}^* = \tilde{V}^{-1}, \quad \tilde{U}^* = \tilde{U}^{-1} \tag{2.8}$$

and

$$A^+_{MN} = N^{-1/2}\Big(\sum_{i=1}^{r} d_i^{-1}v_iu_i^*\Big)M^{1/2}. \tag{2.9}$$

Since

$$N^{-1}A^*MA = N^{-1/2}\Big(\sum_{i=1}^{r} d_i^2 v_iv_i^*\Big)N^{1/2} \tag{2.10}$$