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MULTIGRID MULTI-LEVEL DOMAIN DECOMPOSITION*

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Abstract

The domain decomposition method in this paper is based on PCG(Preconditioned Conjugate Gradient method). If N is the number of subdomains, the number of subproblems solved parallelly in a PCG step is $\frac{4}{3}(1-\frac{1}{4^{\log N+1}})N$. The condition number of the preconditioned system does not exceed $O(1+\log N)^3$. It is completely independent of the mesh size. The number of iterations required, to decrease the energy norm of the error by a fixed factor, is proportional to $O(1+\log N)^{\frac{3}{2}}$.

§1. Triangulation and Subdomain Selection

Let $\Omega\subset R^2$ be a bounded polygonal region, and let

$$\begin{cases} a(u,v)=(f,v), f\in H^{-1}(\Omega), v\in H_0^1(\Omega),\\ u\in H_0^1(\Omega)\end{cases}\tag{1.1}$$

be the variational form of an elliptic operator defined on it. The bilinear form satisfies

$$\begin{cases} a(u,v)=a(v,u),\\ |a(u,v)|\le M'\|u\|\cdot\|v\|,\\ a(u,u)\ge M''\|u\|^2,\end{cases}\tag{1.2}$$

where $\|\cdot\|$ is the Sobolev norm in $H^1(\Omega)$. From (1.2) the norm is equivalent to that introduced by $a(\cdot,\cdot)$ in $H_0^1(\Omega)$. In what follows, we will consider $H_0^1(\Omega)$ as a Hilbert spase with the inner product $a(\cdot,\cdot)$.

We will approximate (1.1) with the finite element method. Triangular partition and linear continuous elements will be used. The triangulation satisfies quasi-uniformity and inverse hypothesis.

1.1. Triangulation. $\mathcal{T}^0$ is a triangulation of Ω satisfying quasi-uniformity and inverse hypothesis. Divide any triangle T^0 of $\mathcal{T}^0$ into four (Fig.1), and we get a partition $\mathcal{T}^1$ with the first refinement. Continue the process with $\mathcal{T}^1$ similarly. After the m-th refinement, we get the final triangulation $\mathcal{T}^m$, which is the partition we really use for finite element approximation. The mesh size of $\mathcal{T}^m$ is h.

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Let $S^m = S_0^h \subset H_0^1(\Omega)$ be the finite element space. $S^0 \subset S^1 \subset \cdots \subset S^m$ are finite element spaces corresponding to 0-level to m-level triangulations. $\hat{\Omega}^l$ represents the set of l-level finite element nodes. $\{\phi_i^l \in S^l, i \in \hat{\Omega}^l\}$ are the usual finite element basis functions.

$$A^l = (a(\phi_i^l, \phi_j^l))_{i,j\in\hat{\Omega}^l} \tag{1.3}$$

is the l-level stiffness matrix, $A^m = A, l = 0, 1, 2, \cdots, m$. $\hat{\Omega} = \hat{\Omega}^m$.

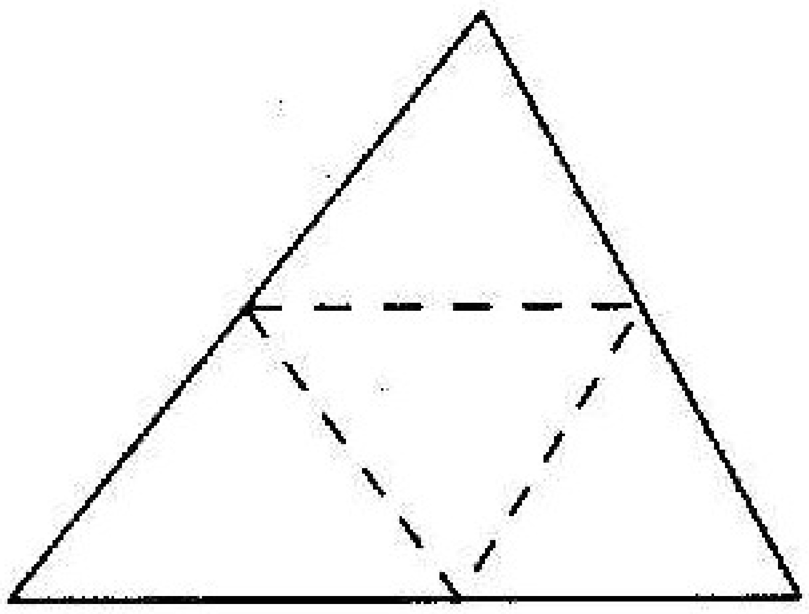

Fig. 1

1.2. Subdomain Selection. $\{\Omega_k^l, l = 0, 1, 2, \cdots, m, k = 1, 2, \cdots, N_l\}$ is a set of open subregions of Ω. For a fixed $l \in \{0, 1, 2, \cdots, m\}$, $\{\Omega_k^l, k = 1, 2, \cdots, N_l\}$ is the set of l-level subdomains.

The following requirements should be met.

A1. $\{\partial\Omega_k^l, k = 1, 2, \cdots, N_l\}$ is a part of the mesh line of the triangulation T^l, $l = 0, 1, 2, \cdots, m$.

A2. For any fixed $l \in \{0, 1, 2, \cdots, m\}$, i.e. on a given level,

$$\bigcup_{k=1}^{N_l} \Omega_k^l = \Omega, \tag{1.4}$$

$\{\Omega_k^l, k = 1, 2, \cdots, N_l\}$ satisfies the quasi-uniformity requirements. H_l is the diameter of l-level subregions, $H_m = H$.

A3. For fixed l, there is another set of subregions $\{\Omega'^l_k, k = 1, 2, \cdots, N_l\}$ so that $\Omega_k^l \subset \Omega'^l_k$ and

$$\mathrm{dist}\{\partial\Omega_k^l \setminus \partial\Omega, \partial\Omega'^l_k \setminus \partial\Omega\} \geq \alpha \cdot H,$$

where α is a fixed constant. At any point in Ω, the number of subregions in $\{\Omega'^l_k, k = 1, 2, \cdots, N_l\}$ which cover this point does not exceed a fixed number (if the coefficient of the function term in the differential operator is strictly positive, this requirement can be released.)

$\hat{\Omega}_k^l = \Omega_k^l \cap \hat{\Omega}^l$ is the set of node points in Ω_k^l. From (1.4) we get

$$\bigcup_{k=1}^{N_l} \hat{\Omega}_k^l = \hat{\Omega}^l. \tag{1.5}$$

It is well known that the relation among the numbers of nodes, triangles and edges of a triangulation is roughly 1 : 2 : 3, and the numbers of triangles of one refined triangulation is four times that of the original one. Therefore,

$$|\hat{\Omega}^l|/|\hat{\Omega}^{l-1}| \simeq 4, \quad l = 1, 2, \cdots, m.$$

To ensure that the scales of subproblems are roughly equal to each other, we require

A4. $N_l = 4^l$, $l = 0, 1, 2, \cdots, m$.

$N = N_m$ is the number of subregions. 0-level to $(m-1)$-level subregions are considered as auxiliary subdomains. The number of levels $m = \log_4 N$ is independent of the mesh size h.