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A NOTE ON CONVERGENCE OF SYMPLECTIC SCHEMES FOR HAMILTONIAN SYSTEMS*[1)]

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Abstract

In this note we prove that all canonical (or symplectic) schemes for Hamiltonian systems constructed in [1-3] are convergent.

In [1-3] Feng and his colleagues proposed a systematical method for the construction of canonical schemes with arbitrary order of accuracy for the Hamiltonian system

$$\frac{dz}{dt} = JH_z, \quad z \in U \subset R^{2n}. \tag{1}$$

In this note we shall prove, using the method in [4], that all these canonical schemes are convergent.

A normal Darboux matrix

$$\begin{aligned} \alpha &= \begin{pmatrix} A_\alpha & B_\alpha \\ C_\alpha & D_\alpha \end{pmatrix} = \begin{pmatrix} J & -J \\ \frac{1}{2}(I+JB) & \frac{1}{2}(I-JB) \end{pmatrix}, \quad B' = B, \\ \alpha^{-1} &= \begin{pmatrix} A^\alpha & B^\alpha \\ C^\alpha & D^\alpha \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(JBJ-J) & I \\ \frac{1}{2}(JBJ+J) & I \end{pmatrix} \end{aligned} \tag{2}$$

define a linear transformation in the product space $R^{2n} \times R^{2n}$ by

$$\begin{pmatrix} \hat{w} \\ w \end{pmatrix} = \alpha \begin{pmatrix} \hat{z} \\ z \end{pmatrix}, \quad \begin{pmatrix} \hat{z} \\ z \end{pmatrix} = \alpha^{-1} \begin{pmatrix} \hat{w} \\ w \end{pmatrix},$$

i.e.

$$\hat{w} = J\hat{z} - Jz, \quad w = \frac{1}{2}(I+JB)\hat{z} + \frac{1}{2}(I-JB)z \quad B' = B. \tag{3}$$

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Let $z \to \hat{z} = g(z,t)$ be the phase flow of the Hamiltonian system (1); it is a time-dependent canonical map. There exist , for sufficiently small $|t|$ and in (some neighborhood of) R^{2n}, a time-dependent gradient map $w \to \hat{w} = f(w,t)$ with Jacobian $f_w(w,t) \in sm(2n)$ (i.e. everywhere symmetric) and a time-dependent generating function $\phi = \phi_{\alpha,H}(w,t)$ such that

$$f(w,t) = \nabla\phi_{\alpha,H}(w,t), \quad A_\alpha g(z,t) + B_\alpha z \equiv (\nabla\phi)(C_\alpha g(z,t) + D_\alpha z, t). \tag{4}$$

On the other hand, for a given time-dependent scalar function $\psi(w,t) : R^{2n} \times R \to R$, we can get a time-dependent canonical map $\tilde{g}(z,t)$. If $\psi(w,t)$ approximates the generating function $\phi_{\alpha,H}(w,t)$ of the Hamiltonian system (1), then $\tilde{g}(z,t)$ approximates the phase flow $g(z,t)$.

For sufficiently small $\tau > 0$ as the time-step, define

$$\psi^{(m)} = \sum_{k=1}^{m} \phi^{(k)}(w)\tau^k, \tag{5}$$

where $\phi^{(1)}(w) = -H(w)$, and for $k \geq 0$, $A^\alpha = \frac{1}{2}(JBJ - J)$,

$$\phi^{(k+1)}(w) = \frac{-1}{k+1}\sum_{m=1}^{k}\frac{1}{m!}\sum_{i_1,\cdots,i_n=1}^{2n} H_{z_{i_1}\cdots z_{i_m}}(w) \times \sum_{\substack{j_1+\cdots+j_m=k \\ j_l \geq 1}} (A^\alpha\nabla\phi^{(j_1)}(w))_{i_1}\cdots(A^\alpha\nabla\phi^{(j_m)}(w))_{i_m}. \tag{6}$$

Then, $\psi^{(m)}(w,\tau)$ is the m-th approximant of $\phi_{\alpha,H}(w,\tau)$, and the gradient map

$$w \to \hat{w} = \tilde{f}(w,\tau) = \nabla\psi^{(m)}(w,\tau) \tag{7}$$

defines a canonical map $z \to \hat{z} = \tilde{g}(z,\tau)$ implicitly by equation

$$A_\alpha\hat{z} + B_\alpha z = (\nabla\psi^{(m)})(C_\alpha\hat{z} + D_\alpha z, \tau). \tag{8}$$

An implicit canonical difference scheme

$$z = z^k \to \hat{z} = z^{k+1} = \tilde{g}(z^k,\tau) \tag{9}$$

for system (1) is obtained in [1-3], and this scheme is of m-th order of accuracy [5]. For the sake of simplicity, we denote $\tilde{g}_\tau(z) = \tilde{g}(z,\tau)$. Then,

$$\tilde{g}_0(z) = z, \qquad \left.\frac{d^i\tilde{g}_\tau(z)}{d\tau^i}\right|_{\tau=0} = \left.\frac{d^i g_\tau(z)}{d\tau^i}\right|_{\tau=0}, \tag{10}$$

where $g_\tau(z)$ is the phase flow $g(z,\tau)$.

Theorem. *If H is analytical in $U \subset R^{2n}$, then scheme (9) is convergent with m-th order of accuracy.*