Journal of Computational Mathematics, Vol.9, No.1, 1991, 41–56.

ON STABILITY AND CONVERGENCE OF THE FINITE DIFFERENCE METHODS FOR THE NONLINEAR PSEUDO-PARABOLIC SYSTEM*

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Abstract

In this paper, we deal with the finite difference method for the initial boundary value problem of the nonlinear pseudo-parabolic system

$$(-1)^M u_t + A(x,t,u,u_x,\ldots,u_{x^{2M-1}})u_{x^{2M}t} = F(x,t,u,u_x,\ldots,u_{x^{2M}}),$$
$$u_{x^k}(0,t) = \psi_{0k}(t),\ u_{x^k}(L,t) = \psi_{1k}(t), k = 0,1,\cdots,M-1,\ u(x,0) = \phi(x)$$

in the rectangular domain $D = [0 \le X \le L, 0 \le t \le T]$, where $u(x,t) = (u_1(x,t), u_2(x,t), \cdots, u_m(x,t))$, $\phi(x)$, $\psi_{0k}(t)$, $\psi_{1k}(t)$, $F(x,t,u,u_x,\ldots,u_{x^{2M}})$ are m-dimensional vector functions, and $A(x,t,u,u_x,\ldots,u_{x^{2M-1}})$ is an $m\times m$ positive definite matrix. The existence and uniqueness of solution for the finite difference system are proved by fixed-point theory. Stability , convergence and error estimates are derived.

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The linear and nonlinear pseudo-parabolic equations and systems often appear in pratical research. There are many works contributed to the finite difference study of different problems for the nonlinear pseudo-parabolic equations. In this paper, we consider the nonlinear pseudo-parabolic system

$$(-1)^M u_t + A(x,t,u,u_x,\ldots,u_{x^{2M-1}})u_{x^{2M}t} = F(x,t,u,u_x,\ldots,u_{x^{2M}}) \tag{1.1}$$

with the nonhomogeneous boundary conditions

$$u_{x^k}(0,t) = \psi_{0k}(t), \quad u_{x^k}(L,t) = \psi_{1k}(t), k = 0,1,\cdots,M-1 \tag{1.2}$$

and the initial condition

$$u(x,0) = \phi(x), \tag{1.3}$$

in the rectangular domain $D = [0 \le x \le L, 0 \le t \le T]$, by the finite difference method, where $u(x,t) = (u_1(x,t), u_2(x,t), \cdots, u_m(x,t))$ and $\phi(x)$, $\psi_{0k}(t)$, $\psi_{1k}(t)$, $F(x,t,u,u_x,\ldots,u_{x^{2M}})$ are m-dimensional vector functions.

The equation for the long waves in nonlinear dispersion

$$u_t + f(u)_x = u_{xxt}$$

* Received July 31, 1987.

is a simple special case of the system (1.1). Some evolutional equations of Sobolev-Galpern type also belong to the system (1.1).

We make the following assumptions for the system (1.1)-(1.3):

(1) The system (1.1)-(1.3) has a unique smooth solution.

(2) $A(x,t,p_0,p_1,\dots,p_{2M-1})$ is an $m\times m$ symmetric positive definite matrix for $(x,t)\in D$ and $p_0,p_1,\dots,p_{2M-1}\in \mathbb{R}^m$. Or rather, there exists a constant $a_0>0$, such that for all $\xi\in\mathbb{R}^m$,

$$(A\xi,\xi)\geq a_0|\xi|^2.$$

(3) $A(x,t,p_0,p_1,\dots,p_{2M-1})$ is a continuous function with respect to $(x,t)\in D$ and it has the first order continuous partial derivatives with respect to $p_0,p_1,\dots,p_{2M-1}\in\mathbb{R}^m$ and with respect to $t\in[0,T]$.

(4) $F(x,t,p_0,p_1,\dots,p_{2M})$ is a continuous function with respect to $(x,t)\in D$ and it has the first order continuous partial derivatives with respect to $p_0,p_1,\dots,p_{2M}\in\mathbb{R}^m$.

(5) $\psi_{ok}(t)\in C^{(1)}([0,T]),\psi_{1k}(t)\in C^{(1)}([0,T])$ and $\phi(x)\in C^{(2M)}([0,L])$ satisfy the following conditions:

$$\psi_{0k}(0)=\phi_{x^k}(0),\quad \psi_{1k}(0)=\phi_{x^k}(L),\quad k=0,1,\cdots,M-1.$$

Let us divide the rectangular domain D into small grids by the parallel lines $x=x_j$ $(j=0,1,\cdots,J)$ and $t=t^n$ $(n=0,1,\cdots,N)$, where $x_j=jh,t^n=n\tau,Jh=L,N\tau=T$ $(j=0,1,\cdots,J;\ n=0,1,\cdots,N)$. Denote the vector valued discrete function on the grid point (x_j,t^n) by v_j^n $(j=0,1,\cdots,J;\ n=0,1,\cdots,N)$. For simplicity, we adopt the same notations and abbreviatons as used in [1-3].

Let us construct the finite difference system

$$(-1)^M\frac{v_j^{n+1}-v_j^n}{\tau}+A_j^{n+\alpha}(v)\frac{\Delta_+^M\Delta_-^M(v_j^{n+1}-v_j^n)}{\tau h^{2M}}=F_j^{n+\alpha}(v),$$
$$j=M,M+1,\cdots,J-M;\ n=0,1,\cdots,N-1 \tag{1.4}$$

where

$$A_j^{n+\alpha}(v)=A(x_j,t^{n+\alpha},\bar\delta^0v_j^{n+\alpha},\bar\delta^1v_j^{n+\alpha},\cdots\cdots,\bar\delta^{M-1}v_j^{n+\alpha},\tilde\delta^Mv_j^{n+\alpha},\cdots\cdots,\tilde\delta^{2M-1}v_j^{n+\alpha}),$$
$$F_j^{n+\alpha}(v)=F(x_j,t^{n+\alpha},\hat\delta^0v_j^{n+\alpha},\hat\delta^1v_j^{n+\alpha},\cdots\cdots,\hat\delta^{M-1}v_j^{n+\alpha},\tilde\delta^Mv_j^{n+\alpha},\cdots\cdots,\tilde\delta^{2M}v_j^{n+\alpha}),$$
$$\bar\delta^kv_j^{n+\alpha}=\sum_{i=j-M}^{j+M-k}(\alpha\beta_{ki}^{(1)}\frac{\Delta_+^kv_i^{n+1}}{h^k}+\beta_{ki}^{(2)}\frac{\Delta_+^kv_i^n}{h^k}),k=0,1,\cdots,M-1,$$
$$\hat\delta^kv_j^{n+\alpha}=\sum_{i=j-M}^{j+M-k}(\alpha\beta_{ki}^{(3)}\frac{\Delta_+^kv_i^{n+1}}{h^k}+\beta_{ki}^{(4)}\frac{\Delta_+^kv_i^n}{h^k}),k=0,1,\cdots,M-1,$$
$$\sum_{i=j-M}^{j+M-k}(\alpha\beta_{ki}^{(1)}+\beta_{ki}^{(2)})=1,\ \sum_{i=j-M}^{j+M-k}(\alpha\beta_{ki}^{(3)}+\beta_{ki}^{(4)})=1,k=0,1,\cdots,M-1,$$
$$v_j^{n+\alpha}=\alpha v_i^{n+1}+(1-\alpha)v_i^n,0\leq\alpha\leq 1,$$
$$\tilde\delta^kv_j^{n+\alpha}=\sum_{i=j-M}^{j+M-k}\tilde\beta_{ki}\frac{\Delta_+^kv_i^{n+1}}{h^k},\ \sum_{i=j-M}^{j+M-k}\tilde\beta_{ki}=1,k=M,M+1,\cdots,2M. \tag{1.5}$$