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SUPERCONVERGENCE OF FEM FOR SINGULAR SOLUTION*

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Superconvergence of the finite element method (FEM) has been discussed extensively for the problem having smooth solution (See Krizek and Neittaanmaki [8]). A typical result in this direction is the following (see Lin and Xie [4] for details). Consider the model problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \boldsymbol{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$ and f is a smooth function. In order to keep the mesh varying regularly we impose on Ω a kind of "piecewise almost uniform triangulation" which can be constructed piecewisely by the vertices of a smoothly transformed uniform mesh. For any node z in the interior of each piece there exist two triangles e and e' such that $e \cap e' = \{z\}$. Then, the average gradient

$$\bar{\nabla} u^h(z) = \frac{1}{2}(\nabla u^h|_e + \nabla u^h|_{e'})$$

has not only the usual type of superconvergence

$$(\bar{\nabla} u^h - \nabla u)(z) = O(h^2)$$

but also an extrapolation type of superconvergence

$$\frac{1}{3}\bar{\nabla}(4u^{h/2} - u^h)(z) - \nabla u(z) = O(h^4 \log \frac{1}{h}).$$

We are concerned in this paper with the superconvergence for the singular solution due to re-entrant corners or changing the boundary conditions.

For simplicity we suppose that Ω is composed of rectangles and the boundary $\partial\Omega$ is parallel to the x-and y-axis and has only one re-entrant corner at the origin 0. Let α be the interior angle at 0 and $\beta = \pi/\alpha$.

It is easy to see that

$$u \in H^3_{(\tau+1)} \quad \text{for } \tau > 1 - \beta,$$

where the Sobolev space $H^3_{(\tau+1)}$ is defined using the weighted norm

$$\|u\|_{3,(\tau+1)} = \Big[\sum_{|j|\le 3}\int_\Omega (|X|^{\tau-2+|j|}|\partial^j u|)^2 dX\Big]^{1/2}$$

with $X = (x, y)$.

We now introduce a rectangular mesh $T^h = \{e\}$, where (x_e, y_e) denotes the center of the element e and $2h_e$ and $2k_e$ are its widths in the x- and y-direction, respectively. Further, we set

$$d_e = \max(h_e, k_e), \quad h = \max\{d_e, e \in T^h\},$$

$$d_0 = \max\{d_e, e \in T^h, 0 \in e\}, \quad r_e = \min\{|X|, X \in e\}.$$

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Let T^h be split into two parts,

$$\Omega_0 = \{e \in T^h, r_e < d_0\}, \quad \Omega_1 = \{e \in T^h, \quad d_0 \le r_e\},$$

where the local meshes are assumed to satisfy the grading conditions

$$d_0 \le ch^q, \quad q > \frac{t}{\beta}, \quad t \le 2;$$

$$c_1 h r_e^p \le d_e \le c h r_e^p, \quad \forall d_e \le r_e, \quad p = 1 - \frac{1}{q},$$

where q is the grading parameter and t is the superconvergence parameter. For example, if $\Omega = (-1,1) \times (-1,1) \setminus [0,1] \times \{0\}$ (a slit domain), such meshes can be constructed by taking nodes

$$(\pm(i/n)^q, \pm(j/n)^q)(1 \le i, j \le n), \quad q > 2t.$$

Since a larger t will lead to a larger q, the user has to make up his choice between a higher accuracy and a less graded mesh. We note that the total number of nodes of the graded meshes is the same as for a uniform mesh of size h, and that the size of the largest element is of the order h.

Let

$$\Omega_2 = \{X \in \Omega, \quad |X| \ge \rho > 0\},$$

z be the interior node of Ω_2 and N the number of all interior nodes of Ω_2:

$$N = O(h^{-2}).$$

For such z there exist two elements e and e' such that $e \cap e' = \{z\}$ and we can define, for $\nu \in S^h$ the piecewise bilinear finite element space, the average gradient

$$\bar\partial_x \nu(z) = \frac{h_e}{h_e + h_{e'}} \partial_x \nu|_{e'}(z) + \frac{h_{e'}}{h_e + h_{e'}} \partial_x \nu|_e(z),$$

$$\bar\partial_y \nu(z) = \frac{k_e}{k_e + k_{e'}} \partial_y \nu|_{e'}(z) + \frac{k_{e'}}{k_e + k_{e'}} \partial_y \nu|_e(z).$$

Let $u^I \in S^h$ be the interpolation of u and $u^R \in S^h$ the Ritz projection of u. It is easy to see from Taylor expansion the superconvergence of u^I after averaging:

Lemma 1.

$$|(\bar\partial u^I - \partial u)(z)| \le ch^t \|u\|_{3,\infty,\Omega_2},$$

where the notation $\bar\partial$ means $\bar\partial_x$ or $\bar\partial_y$.

Our purpose is to prove the superconvergence of u^R after averaging:

Theorem. *The grading parameter q increases the gradient accuracy from β-order to nearly $q\beta$-order:*

$$\left[\frac{1}{N} \sum_{z \in \Omega_2} |(\bar\partial u^R - \partial u)(z)|^2\right]^{1/2} \le ch^t, \quad t < q\beta.$$

The proof of our theorem is based on the lemmas as follows (c.f. [1]–[2]).

Lemma 2. *For the function $F(x)$ satisfying $F(x_e \pm h_e) = 0$, we have*

$$\int_{x_e - h_e}^{x_e + h_e} F dx = \frac{1}{2} \int_{x_e - h_e}^{x_e + h_e} P F'' dx,$$

where $P(x) = (x - x_e + h_e)(x - x_e - h_e)$.