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RECURRENCE RELATIONS FOR THE COEFFICIENTS IN ULTRASPHERICAL SERIES SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract

A method is presented for obtaining recurrence relations for the coefficients in ultraspherical series of linear differential equations. This method applies Doha's method (1985) to generate polynomial approximations in terms of ultraspherical polynomials of $y(zx), -1 \le x \le 1, z \in \underset{\sim}{C}, |z| \le 1$, where y is a solution of a linear differential equation. In particular, rational approximations of $y(z)$ result if x is set equal to unity. Two numerical examples are given to illustrate the application of the method to first and second order differential equations. In general, the rational approximations obtained by this method are better than the corresponding polynomial approximations, and compare favourably with Padé approximants.

§1. Introduction

The truncated Chebyshev series has been widely used in numerical analysis as a good numerical approximation to $y \in C[-1,1]$ using the supremum norm

$$\|y\|_\infty = \sup_{x\in[-1,1]} |y(x)|.$$

Lanczos[1] and Handscomb[2] have compared the performance of truncated Chebyshev series with truncated ultraspherical expansions. Light[3-4] has investigated conditions under which approximation to continuous functions on $[-1,1]$ by series of Chebyshev polynomials is superior to approximation by other ultraspherical orthogonal expansions. In particular, he has derived conditions on the Chebyshev coefficients which guarantee that the Chebyshev expansion of the corresponding functions converges more rapidly than expansions in Legendre polynomials or Chebyshev polynomials of the second kind.

As candidates for the efficient representation of mathematical functions by easily computed expressions, rational functions are often to be preferred to polynomials. Indeed it has been found empirically that, in general, rational approximations can achieve a smaller maximum error for the same amount of computation than polynomial approximations; see for instance, Ralston and Rabinowitz[5].

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The well-known effective means of producing rational approximations to a function $y(z)$ in a complex variable is to develop elements of Padé table from its Taylor series. This table is a two-dimensional array whose (m,n) element is defined as that rational function of degree m in the numerator and n in the denominator whose own Taylor series expansion, $\sum_{r=0}^{\infty} c_r z^r$ say, agrees with that of $y(z)$ up to and including the term in z^{m+n}. Discussion of Padé table and its convergence is considered in Bender and Orszag [6].

In Doha [7], a method for obtaining simultaneously polynomial and rational approximations from Chebyshev and Legendre series for functions defined by linear differential equations with its associated boundary conditions has been described.

Our principal aim in the present paper is twofold:

(i) to give an extension of Doha's method, but the function $y(x)$ and its derivatives are expanded in ultraspherical polynomials $C_n^{(\alpha)}$. This extension, however, could be useful in applications.

(ii) to compare computationally the performance of the rational approximation obtained from Chebyshev series of the first kind and those obtained from the ultraspherical series for varying α.

The ultraspherical expansions are defined by

$$y(x) = \sum_{n=0}^{\infty} a_n C_n^{(\alpha)}(x)$$

where the coefficients a_n are given by

$$a_n = \int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}} y(x) C_n^{(\alpha)}(x)dx \Big/ \int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}} \{C_n^{(\alpha)}(x)\}^2 dx \tag{1}$$

and $C_n^{\alpha)}(x)$ satisfy the orthogonality relation

$$\int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)}(x) C_m^{(\alpha)}(x)dx = 0, \quad m \neq n, \quad \alpha > -\frac{1}{2}.$$

For our present purposes it is convenient to standardize the ultraspherical polynomials so that

$$C_n^{(\alpha)}(1) = \Gamma(n+2\alpha)/\Gamma(2\alpha)n!$$

In this form the polynomials may be generated using the recurrence formula

$$(n+1)C_{n+1}^{(\alpha)}(x) = 2(n+\alpha)xC_n^{(\alpha)}(x) - (n+2\alpha-1)C_{n-1}^{(\alpha)}(x), \quad n = 1,2,3,\cdots$$

starting from $C_0^{(\alpha)}(x) = 1$ and $C_1^{(\alpha)}(x) = 2\alpha x$, or obtained from Rodrigue's formula

$$C_n^{(\alpha)}(x) = \frac{(-1)^n \Gamma(n+2\alpha)\Gamma(n+1/2)}{2^n n! \Gamma(2\alpha)(n+\alpha+1/2)} (1-x^2)^{\frac{1}{2}-\alpha} D^n[(1-x^2)^{n+\alpha-\frac{1}{2}}]$$

where $D \equiv \frac{d}{dx}$. Certain values of α correspond to more familiar sets of orthogonal polynomials, the Legendre polynomials given by $C_n^{(1/2)}(x) = P_n(x)$, the Chebyshev polynomials of the second kind by $C_n^{(1)}(x) = U_n(x)$ and the Chebyshev polynomials of the first kind by

$$T_n(x) = \frac{n}{2} \lim_{\alpha \to 0} \frac{1}{\alpha} C_n^{(\alpha)}(x).$$