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NONLINEAR STABILITY OF GENERAL LINEAR METHODS*[1)]

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Abstract

This paper is devoted to a study of stability of general linear methods for the numerical solution of nonlinear stiff initial value problems in a Hilbert space. New stability concepts are introduced. A criterion of weak algebraic stability is established, which is an improvement and extension of the existing criteria of algebraic stability.

§1. Introduction

The main goal of this paper is to make further advances on the theories of algebraic stability for general linear methods (cf. [1–4]). In Section 2, we introduce a family of classes of nonlinear test problems, $\{K_{\sigma,\tau} : \sigma\tau < 1\}$, in a Hilbert space. Section 3 is concerned with general linear methods in brief. In Section 4 a series of new stability concepts is introduced. In Section 5 we establish the criterion of (k,p,q)-weak algebraic stability, which is an essential improvement and extension of the criteria of algebraic stability presented by Burrage and Butcher[4].

§2. Test Problems

Let X be a real or complex Hilbert space with the inner product $\langle \cdot\ ,\ \cdot \rangle$ and the corresponding norm $\|\cdot\|$, D an infinite subset of X, and $f : [0,+\infty) \times D \to X$ a given mapping. Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \geq 0, & (2.1) \\ y(0) = \iota, \qquad \iota \in D, & (2.2) \end{cases}$$

which is assumed to have a unique solution $y(t)$ on the interval $[0, +\infty)$.

Definition 1. *Let σ, τ be real constants with $\sigma\tau < 1$. Then the class of all problems* (2.1)–(2.2) *with*

$$2Re\ \langle u - v, f(t,u) - f(t,v)\rangle \leq \sigma\|u - v\|^2 + \tau\|f(t,u) - f(t,v)\|^2 \quad \forall u, v \in D, \quad t \geq 0 \quad (2.3)$$

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is called the class $K_{\sigma,\tau}$.

From Definition 1 we obtain the following propositions:

Proposition 1. If $\sigma < 0, \varepsilon \geq 0$, then $K_{\sigma,\tau} \subset K_{\sigma-\varepsilon,\tau+\varepsilon M(\sigma,\tau)}$, where

$$M(\sigma,\tau) = [(1+\sqrt{1-\sigma\tau})/\sigma]^2. \tag{2.4}$$

Proposition 2. If $\tau < 0, \varepsilon \geq 0$, then $K_{\sigma,\tau} \subset K_{\sigma+\varepsilon N(\sigma,\tau),\tau-\varepsilon}$, where

$$N(\sigma,\tau) = [(1+\sqrt{1-\sigma\tau})/\tau]^2. \tag{2.5}$$

Proposition 3. If X is the usual N-dimensional complex U-space and for fixed t, $\psi(t) \in X, A(t)$ is an $N \times N$ complex matrix, then the linear system

$$y'(t) = A(t)y(t) + \psi(t), \quad t \geq 0; \quad y(0) = \iota, \quad \iota \in D = X \tag{2.6}$$

belongs to the class $K_{\sigma,\tau}$ if and only if $\sigma \geq \sup_{t\geq 0} \lambda_{\max}$, where $\lambda_{\max}$ denotes the greatest eigenvalue of the Hermite matrix $A^* + A - \tau A^* A$.

In the literature only the special cases $K_{0,0}, K_{\sigma,0}$ and $K_{0,\tau}$ have been used respectively as the test problem class (cf. [1–5]).

§3. General Linear Methods

Consider the general linear method for solving (2.1)–(2.2):

$$\begin{cases} Y_i^{(n)} = h\sum_{j=1}^{s} c_{ij}^{11} f(T_j^{(n)}, Y_j^{(n)}) + \sum_{j=1}^{r} c_{ij}^{12} y_j^{(n-1)}, & i = 1,2,\cdots,s; \\ y_i^{(n)} = h\sum_{j=1}^{s} c_{ij}^{21} f(T_j^{(n)}, Y_j^{(n)}) + \sum_{j=1}^{r} c_{ij}^{22} y_j^{(n-1)}, & i = 1,2,\cdots,r; \\ y_n = \sum_{j=1}^{r} \beta_j y_j^{(n)}, \end{cases} \tag{3.1}$$

where $h > 0$ is the stepsize, c_{ij}^{IJ} and β_j are constants in the base field of $X, Y_i^{(n)}, y_i^{(n)}$ and y_n are approximations to $y(T_i^{(n)}), H_i(t_i^{(n)})$ and $y(t_n + \eta h)$ respectively, $T_i^{(n)} = t_{n-1} + \mu_i h, t_i^{(n)} = t_n + \nu_i h, t_n = nh, \mu_i, \nu_i, \eta$ are nonnegative constants, and each $H_i(t_i^{(n)})$ is a piece of information about $y(t)$.

Throughout this paper we always assume that each $y_i^{(n)}$ is an approximation to $y(t_i^{(n)})$ for $i \leq l$ and the following equalities hold exactly:

$$y_{l+i}^{(n)} = hf(t_i^{(n)}, y_i^{(n)}), \quad i = 1,2,\cdots,l; \quad n = 0,1,2,\cdots, \tag{3.2}$$

where l is a fixed nonnegative integer not greater than $r/2$.

For any given $M \times N$ matrix $A = [a_{ij}]$ we define a linear mapping $\widetilde{A} : X^N \to X^M$ such that for any $U = (u_1, u_2, \cdots, u_N) \in X^N$ with each $u_i \in X$,

$$\widetilde{A}U = V = (v_1, v_2, \cdots, v_M) \in X^M,$$