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ON THE NUMBER OF ZEROES OF EXPONENTIAL SYSTEMS[*1)]

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Abstract

A system $E: C^n \to C^n$ is said to be an exponential one if its terms are $ae^{im_1 Z_1} \cdots e^{im_n Z_n}$. This paper proves that for almost every exponential system $E: C^n \to C^n$ with degree $(q_1, \cdots, q_n)$, E has exactly $\Pi_{j=1}^n(2q_j)$ zeroes in the domain

$$D = \{(Z_1, \cdots, Z_n) \in C^n : \ Z_j = x_j + iy_j, \ x_j, \ y_j \in R, \ 0 \le x_j < 2\pi, \ j = 1, \cdots, n\},$$

and all these zeroes can be located with the homotopy method.

§1. Introduction

Let E: $C^n \to C^n$ be an exponential system, where C^n is the n-dimensional complex space. By an exponential system, we mean that each term in every equation is of the form

$$ae^{im_1 Z_1} \cdots e^{im_n Z_n}, \tag{1.1}$$

where $i = \sqrt{-1}$, a is a complex number, Z_j a complex variable, and m_j an integer. For each term in each equation, consider the sum $|m_1| + \cdots + |m_n|$. Let q_j be the maximum sum in equation j. We assume $q_j > 0$ for all j. We call q_j the degree of E_j, and $(q_1, \cdots, q_n)$ the degree of the system E. In this paper, let

$$D = \{(Z_1, \cdots, Z_n) \in C^n : \ Z_j = x_j + iy_j, \ x_j, \ y_j \in R, \ 0 \le x_j < 2\pi, \ j = 1, \cdots, n\},$$

Let E: $C^n \to C^n$ be given as above. Now, we distinguish certain coefficients of E. Let a_{kj} be the coefficient of term $e^{iq_k Z_j}$ in E_k, and b_{kj} the coefficient of the term $e^{-iq_k Z_j}$ in E_k for $k, j = 1, \cdots, n$. Let $A = ((a_{kj})|(b_{kj})) \in C^{2n^2}$. Define B to be the other coefficients of the terms with degree q_k in E_k for all $k = 1, \cdots, n$. Let a_i be the constant term of E_i for $i = 1, \cdots, n$ and $a = (a_1, \cdots, a_n) \in C^n$. Let b be all coefficients of E other than a, A and B. Then (a, A, b, B) uniquely defines E. We write E as $E(\cdot, a, A, b, B)$.

Utilizing homotopy methods, this paper studies zero distribution of exponential systems. Section 2 discusses numbers of the zeroes of the systems. Section 3 applies the results to triangular polynomial systems. Section 4 explores the relationship between exponential systems and polynomial systems, and points out that it is unreasonable to transform exponential systems into corresponding polynomial systems for the purpose of locating all isolated zeroes. Section 5 contains several numerical examples.

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§2. Main result

Lemma 1[1]. *Let $H : R^n \times R^m \to R^p$ be a smooth mapping. Suppose 0 is a regular value of H. Then for almost all $a \in R^m$, 0 is a regular value of $H(\cdot, a) : R^n \to R^p$.*

Lemma 2[1]. *Suppose $F : C^n \to C^n$ is an analytic mapping. Regard F as a real mapping $F : R^{2n} \to R^{2n}$ in the way of identifying $(Z_1, \cdots, Z_n)$ with $(x_1, y_1, \cdots, x_n, y_n)$, where $Z_j = x_j + iy_j$, $i = \sqrt{-1}$, $x_j, y_j \in R$, for $j = 1, \cdots, n$. Then the real Jacobian determinant $\det \partial F/\partial(x_1, y_1, \cdots, x_n, y_n)$ is nonnegative everywhere. Furthermore, if 0 is a regular value of F, then the determinant is positive in $F^{-1}(0)$.*

Lemma 3[2]. *Let $H : R^n \times [0,1] \to R^n$ be a smooth mapping. Suppose 0 is a regular value of H. Then for any curve $\lambda(s) = (x(s), t(s))$ in $H^{-1}(0)$,*

$$\text{sgn } \dot{t}(s) = \text{sgn det } \frac{\partial H}{\partial x}(\lambda(s)) \quad \textit{for all } s,$$

or

$$\text{sgn } \dot{t}(s) = -\text{sgn det } \frac{\partial H}{\partial x}(\lambda(s)) \quad \textit{for all } s,$$

where s is the arc length.

Let E be an exponential system with degree $(q_1, \cdots, q_n)$, and define an auxiliary mapping $E_0 = (E_{01}, \cdots, E_{0n}) : C^n \to C^n$ by

$$E_{0j}(Z) = e^{iq_jZ_j} + e^{-iq_jZ_j}, \text{ for } j = 1, \cdots, n.$$

It is clear that E_0 has exactly $\Pi_{j=1}^n(2q_j)$ zeroes in D and 0 is a regular value of E_0.

Define homotopy $H : C^n \times [0,1] \to C^n$ by

$$H(Z,t) = tE(Z) + (1-t)E_0(Z). \tag{2.1}$$

Then $H(\cdot, 0) = E_0(\cdot)$ and $H(\cdot, 1) = E(\cdot)$. The following lemma is direct from Lemma 1.

Lemma 4. *Assume H as in (2.1). Then for all A, b and B, and for almost all $a \in C^n$, 0 is a regular value of H.*

We say H is regular if 0 is a regular value of H. Fix $a \in C^n$ such that H is regular. Then $H^{-1}(0)$ is a one-dimensional manifold. By Lemmas 2 and 3, for any curve $\lambda(s) = (Z(s), t(s))$ of $H^{-1}(0)$, $t(s)$ is a monotone function of s. So we can write $\lambda(s)$ as $\lambda(t) = (Z(t), t), 0 \leq t \leq 1$. Hence, we have

Lemma 5. *Assume H as above. Then $H^{-1}(0)$ consists of four kinds of curves as follows (shown in* Fig.1*):*

(1) *curves of finite lengths starting at $C^n \times \{0\}$ and ending at $C^n \times \{1\}$;*
(2) *unbounded curves with only one boundary point in $C^n \times \{0\}$;*
(3) *unbounded curves with only one boundary point in $C^n \times \{1\}$;*
(4) *unbounded curves in $C^n \times (0,1)$.*

Now, we prove that for almost all $A \in C^{2n^2}$, $H^{-1}(0)$ is bounded. First, we give some definitions. Let E be an exponential system with degree $(q_1, \cdots, q_n)$. Let $s = (s_1, \cdots, s_n) \in \{1, -1\}^n$. Define $\bar{Z}_j = e^{is_jZ_j}$ for $j = 1, \cdots, n$. Then $E(Z, a, A, b, B)$ becomes a mapping $E_s(\bar{Z}, a, A, b, B)$ that consists of the terms like $a\bar{Z}_1^{m_1} \cdots \bar{Z}_n^{m_n}$. Let $PE_s = (PE_{s1}, \cdots, PE_{sn})$ be the polynomial part of $E_s(\cdot, a, A, b, B)$. That is, $PE_{sk}(\bar{Z})$ consists of all polynomial terms like $a\bar{Z}_1^{m_1} \cdots \bar{Z}_n^{m_n} (m_j \geq 0$ for $j = 1, \cdots, n)$ in $PE_{sk}(\bar{Z})$, $k = 1, \cdots, n$. We call PE_s the polynomial system of E with respect to s. It is clear that the degree of PE_s is $(q_1, \cdots, q_n)$. Since the number of the elements of $\{1, -1\}^n$ is 2^n, we have 2^n different polynomial systems PE_s, each of which has a different s.