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A PSEUDOSPECTRAL METHOD FOR SOLVING NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, we propose a new kind of pseudospectral schemes with a restraint operator to solve the periodic problem of Navier-Stokes equations. The generalized stability of the schemes is analysed and the convergence is proved. Numerical results are presented also.

§1. Introduction

We consider the periodic problem of Navier-Stokes equations as follows:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + (U(x,t)\cdot\nabla)U(x,t) + \nabla P(x,t) - \nu\nabla^2 U(x,t) = f(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t)\in\Omega\times(0,T], \\ \nabla\cdot U(x,t) = 0, \quad (x,t)\in\Omega\times[0,T], \\ U(x,0) = U_0(x), \quad P(x,0) = P_0(x), \quad x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega=[0,2\pi]^n, n=2$ or 3, $\nu\geq 0$. $U=(U_1,\cdots,U_n)$ is the velocity. P is the ratio of pressure to density. The functions U_0, P_0 and f are given with the period 2π for all the space variables. We require in addition

$$\int_\Omega P(x,t)dx = 0, \qquad \forall t\in[0,T].$$

It is well known that if the genuine solutions of PDEs are infinitely differentiable, then the convergence rates of their spectral/pseudospectral approximations are infinite[1]. Hence they have been widely used in computational fluid dynamics[2]. The pseudospectral methods are easier to implement. But they are not as stable as the spectral ones due to "aliasing". Therefore some authors proposed the filtering techniques[3–5]. In this paper, a new kind of pseudospectral schemes with a restraint operator is constructed to solve (1.1). The generalized stability and the convergence are analysed. In particular, the uniform stability and convergence (independent of the coefficient ν) are obtained in some cases.

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§2. Notations and Lemmas

Denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm of $L^2(\Omega)$ respectively. Let $\|\cdot\|_\infty$ be the norm of $L^\infty(\Omega)$. $\|\cdot\|_\mu$ and $|\cdot|_\mu$ denote the norm and semi-norm of $H^\mu(\Omega)$. Define

$$C_p^\infty(\Omega) = \{u/u \in C^\infty(\Omega), u \text{ has the period } 2\pi \text{ for the variables } x_q, 1 \le q \le n\},$$

and let $H_p^\mu(\Omega)$ be the closure of $C_p^\infty(\Omega)$ in $H^\mu(\Omega)$. Let X be a Banach space. Define

$$L^2(0,T;X) = \{u/u : [0,T] \to X, \|u\|_{L^2(0,T;X)} = \left(\int_0^T \|u(t)\|_X^2 dt\right)^{1/2} < \infty\},$$

$$C(0,T;X) = \{u/u : [0,T] \to X \text{ is strongly continuous}, |||u|||_X = \max_{0\le t\le T} \|u(t)\|_X\}.$$

Denote by Z the set of integers. For $k = (k_1, \cdots, k_n) \in Z^n$, let $|k|_\infty = \max_{1\le q\le n} |k_q|$ and $|k| = \left(\sum_{q=1}^n k_q^2\right)^{1/2}$. For a positive integer N, we define

$$V_N = \mathrm{Span}\{e^{ik\cdot x}/k \in Z^n, |k|_\infty \le N\}, \quad W_N = \mathrm{Span}\{e^{ik\cdot x}/k \in Z^n, |k| \le N\}.$$

Let P_N be the orthogonal projection operator from $L^2(\Omega)$ onto W_N. $\tilde{P}_c$ is the Lagrange interpolation operator from $C(\Omega)$ onto V_N at points $x^{(j)} = 2\pi j/(2N+1)$, $j \in Z^n, |j|_\infty \le N$. Let $P_c = P_N \tilde{P}_c$. It is easy to see that[5]

$$(P_c(uv), w) = (P_c(wv), u), \quad \forall u, v, w \in W_N. \tag{2.1}$$

Lemma 2.1[3]. *If $u, v \in V_N$, then*

$$\text{(i)} |u|_1^2 \le nN^2\|u\|^2, \tag{2.2}$$

$$\text{(ii)} |uv|_1^2 \le n(2N+1)^n(\|u\|^2|v|_1^2 + \|v\|^2|u|_1^2). \tag{2.3}$$

Lemma 2.2[6]. *If $0 \le \mu \le \sigma$, then for all $u \in H_p^\sigma(\Omega)$,*

$$\|P_N u - u\|_\mu \le CN^{\mu-\sigma}|u|_\sigma.$$

If in addition $\sigma > n/2$, then

$$\|P_c u - u\|_\mu \le CN^{\mu-\sigma}|u|_\sigma.$$

Now we define the restraint operator R_r for $r > 1$, i.e., if

$$u(x) = \sum_{|k|\le N} u_k e^{ik\cdot x}$$

then

$$R_r u(x) = \sum_{|k|\le N} \left(1 - \left(\frac{|k|}{N}\right)^r\right) u_k e^{ik\cdot x}.$$

Lemma 2.3. *If $0 \le \mu \le \sigma, r \ge \sigma - \mu$, then for all $u \in W_N$,*

$$\|R_r u - u\|_\mu \le CN^{\mu-\sigma}|u|_\sigma.$$

Proof. Since $0 \le \mu \le \sigma, r \ge \sigma - \mu$, we have clearly that

$$\|R_r u - u\|_\mu^2 \le C \sum_{|k|\le N} \frac{|k|^{2\sigma}}{N^{2(\sigma-\mu)}} \left(\frac{|k|}{N}\right)^{2r+2(\mu-\sigma)} |u_k|^2 \le CN^{2(\mu-\sigma)}|u|_\sigma^2.$$