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A Global Property of Restarted FOM Algotrithm

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Abstract. In this paper an interesting property of the restarted FOM algorithm for solving large nonsymmetric linear systems is presented and studied. By establishing a relationship between the convergence of its residual vectors and the convergence of Ritz values in the Arnoldi procedure, it is shown that some important information of previous FOM(*m*) cycles may be saved by the iteration approximates at the time of restarting, with which the FOM(*m*) cycles can complement one another harmoniously in reducing the iteration residual. Based on the study of FOM(*m*), two polynomial preconditioning techniques are proposed; one is for solving nonsymmetric linear systems and another is for forming an effective starting vector in the restarted Arnoldi method for solving nonsymmetric eigenvalue problems.

Keywords: nonsymmetric linear systems, nonsymmetric eigenvalue problems, iterative methods, FOM, Arnoldi's method, restarting; polynomial preconditioning.

1. Introduction

The Arnoldi method is an orthogonal projection onto the Krylov subspace defined as

$$K_m(v, A) = \{v, Av, A^2v, \cdots, A^{m-1}v\}$$

for a nonsymmetic matrix $A \in R^{n,n}$. The procedure was introduced in 1951 as a means of reducing a dense matrix into Hessenberg form. Arnoldi introduced this method precisely in this manner and he hinted that the process could give good approximations to some eigenvalues if stopped before completion. It was later discovered that this strategy lead to a good technique for approximating eigenvalues of large sparse matrices, see [9].

Algorithm 1: Arnoldi for eigenvalue problems with restarting

(1) Start: Choose a starting vector and normalize for v_1;

(2) Arnoldi iteration: For $j = 1,2,\cdots,m$ do

$h_{ij} = (v_i, Av_j), \quad i = 1,2,\cdots, j$

$w_j = Av_j - \sum_{i=1}^{j} h_{ij} v_i$

$h_{j+1,j} = \| w_j \|$

$v_{j+1} = w_j / h_{j+1,j}$

(3) Compute approximate eigenpairs: Let H_m be the $m \times m$ upper-Hessenberg matrix whose nonzero entries are defined in the Arnoldi iteration. Let V_m be the $n \times m$ matrix whose columns are v_1 through v_m. Compute eigenpairs $\{\lambda_i^{(m)}, g_i\}$ of H_m as desired. The $\lambda_i^{(m)}$ is a Ritz value and $u_i = V_m g_i$ is the associated Ritz vector.

(4) Restart: Residual norms can be checked for convergence. If needed, choose a new starting vector for v_1 and go to (2).

The Arnoldi method was also extended to solve nonsymmetric linear systems in 1981 by Saad [10]. Consider the linear systems $Ax = b$. The resulting algorithm takes the form:

Algorithm 2: Arnoldi for linear systems with restarting

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(1) Start: Choose x_0 and compute $r_0 = b - Ax_0$. Let $\beta = \| r_0 \|$ and $v_1 = r_0 / \beta$.
(2) Iterate: Perform m steps of the Arnoldi iteration with the starting vector v_1.
(3) Form the approximate solution:

$$x_m = x_0 + V_m y_m,$$

where y_m is the solution of $H_m y = \beta e_1$.
(4) Restart: Compute the residual norm $\| r_m \|$, if satisfied then stop, else let $x_0 = x_m$, $\beta = \| r_m \|$, $v_1 = r_m / \beta$, and go to (2).

The two algorithms described above are closely related. In order for a clear statement, the Arnoldi method for solving linear systems is referred to as the full orthogonalization method (FOM). Restarting is generally needed to reduce storage requirements and orthogonalization costs of the two algorithms. It is one of the most important strategy in successfully carrying out the Arnoldi method. However, restarting leaves many things unknown to us. It seems that some classical results of the unrestarted method can not be directly applied to its restarted scheme. For example, a relationship established in [2,4] shows that the unrestarted FOM can never beat GMRES [12] in term of the convergence of residual norm. However, it is observed that when restarted, FOM(*m*) may yield much rapider convergence than GMRES(*m*) with the same restart frequency *m* in some cases, see [14], [18]. Observations like this indicates that we should present a new view to the restarted Arnoldi method.

In this paper we mainly present a global property of the restarted Arnoldi method for solving nonsymmetric linear systems. By establishing a relationship between the convergence of FOM(*m*) residual vectors and the convergence of Ritz values in the Arnoldi procedure, it will be shown that some important information of previous FOM(*m*) cycles may be saved by the iteration approximates at the time of restarting, with which the FOM(*m*) cycles can complement one another harmoniously in reducing the iteration residual. The same property was first observed for the restarted GMRES algorithm [17], and has attracted wider interest recently [1, 15, 19, 20].

In Section 2 the global property of FOM(*m*) is described. In section 3 two applications of the new property of FOM(*m*) are discussed. In Section 4 some numerical tests are reported. Finally, in Section 4 we draw conclusions.

2. A global property of FOM(*m*)

It is easy to show that the FOM residual r_m is a member of $K_{m+1}(r_0, A)$, and hence it can be written as a *k*th-degree polynomial in *A*, acting on r_0:

$$r_m = p_m(A) r_0,$$

where $p_m(z)$ is known as "FOM residual polynomial", satisfying $p_m(0) = 1$. The following lemma [16, Lemma 1.5] reveals a fundamental relationship between FOM and the corresponding Arnoldi procedure for eigenvalue problems (i.e., both of them start with the same initial vector $v_1 = r_0 / \| r_0 \|$).

Lemma 1. Assume that FOM converges steadily. Then the $\lambda_i^{(m)} \neq 0$ ($i = 1,2,\cdots,m$) and FOM residual polynomial $p_m(z)$ satisfies

$$p_m(z) = \frac{(\lambda_1^{(m)} - z)(\lambda_2^{(m)} - z)\cdots(\lambda_m^{(m)} - z)}{\lambda_1^{(m)} \lambda_2^{(m)} \cdots \lambda_m^{(m)}}. \tag{1}$$

When inserted in FOM(m), the Arnoldi procedure for eigenvalue problems will also be restarted. However, since its starting vectors are taken as the iteration residuals of FOM(m), it is not guaranteed that the Ritz pairs may continuously converge. Nevertheless, what we are really interested in is how closely Ritz values approximate the corresponding eigenvalues in each of the FOM(m) cycles. Then the modulus of $p_m(z)$ on the spectrum of A can be estimated according to (1).

We assume for simplicity that A and H_m are all diagonalizable. Let P_m be the orthogonal projector on $K_m(r_0, A)$ and $\{\varphi_i\}_{i=1,2,\cdots,n}$ be the normalized eigenvectors of A. The following result is an immediate consequence of [5, Theorem 3.7].

Lemma 2. Let $\gamma_m = \| P_m A(I-P_m) \|$, $S_m^{-1} H_m S_m = diag(\lambda_1^{(m)}, \cdots, \lambda_m^{(m)})$ and $cond(S_m) = \| S_m \| \| S_m^{-1} \|$. Assume $\| (I-P_m)\varphi_i \|$ to be small enough. Then there exists an eigenvalue $\lambda_i^{(m)}$ of H_m such that

$$| \lambda_i^{(m)} - \lambda_i | \leq 2\gamma_m cond(S_m) \frac{\| (I-P_m)\varphi_i \|}{\| P_m \varphi_i \|}.$$

Now let us estimate $\| (I-P_m)\varphi_i \|$. The initial vector r_0 is written as $r_0 = \sum_{j=1}^n \alpha_j \varphi_j$. The following lemma was established in [9].

Lemma 3. Assume that $\alpha_i \neq 0$ and let $\xi_i = \sum_{j=1, j\neq i}^n | \alpha_j | / | \alpha_i |$. Then

$$\| (I-P_m)\varphi_i \| \leq \xi_i \min_{\substack{\deg p \leq m-1 \\ p(\lambda_i)=1}} \max_{\substack{j=1,2,\cdots,n \\ j\neq i}} | p(\lambda_j) | \equiv \xi_i \varepsilon_i^{(m)}.$$

Note that for given m, $\varepsilon_i^{(m)}$ is fixed. Then the degree of convergence of Ritz values will be reflected in the scalar ξ_i.

Define "product polynomials" of FOM(m) as

$$\pi_s(z) = p_{m,s}(z) p_{m,s-1}(z) \cdots p_{m,1}(z), \; s = 1,2,\cdots,$$

where $p_{m,j}(z), (j = 1,2,\cdots,s)$ is the residual polynomial of the j th FOM(m) cycle. With the previous preliminaries, we establish the main result below, which is an explicit polynomial characterization of FOM(m).

Theorem 4. Assuming FOM(m) starts with $r_0 = \sum_{j=1}^n \alpha_j \varphi_j$ and $\alpha_i \neq 0$, with a proper order of $\{\lambda_i\}_{i=1}^n$ we will have

$$| \pi_s(\lambda_i) | \leq F_{m,i} \frac{\sum_{j=1, j\neq i}^n | \alpha_j \pi_{s-1}(\lambda_j) |}{| \alpha_i |}, \tag{2}$$

with

$$F_{m,i} = 2 \frac{\prod_{j=1, j\neq i}^m | \lambda_j^{(m)} - \lambda_i |}{\prod_{j=1}^m | \lambda_j^{(m)} |} \gamma_m cond(S_m) \frac{1}{\| P_m \varphi_i \|} \varepsilon_i^{(m)},$$

where the scalars with the subscript or superscript (m) are all associated with the s th cycle of FOM(m).

***Proof*:** It follows from $r_{(s-1)m} = \pi_{s-1}(A) r_0$ that $r_{(s-1)m} = \sum_{j=1}^n \alpha_j \pi_{s-1}(\lambda_j) \varphi_j$. Then

$$\begin{aligned}
| \pi_s(\lambda_i) | &= | p_{m,s}(\lambda_i) \pi_{s-1}(\lambda_i) | \\
&\overset{Lemma\ 1}{=} \frac{\prod_{j=1, j\neq i}^m | \lambda_j^{(m)} - \lambda_i |}{\prod_{j=1}^m | \lambda_j^{(m)} |} | \lambda_i^{(m)} - \lambda_i | \, | \pi_{s-1}(\lambda_i) | \\
&\overset{Lemma\ 2}{\leq} 2 \frac{\prod_{j=1, j\neq i}^m | \lambda_j^{(m)} - \lambda_i |}{\prod_{j=1}^m | \lambda_j^{(m)} |} \gamma_m cond(S_m) \frac{\| (I-P_m)\varphi_i \|}{\| P_m \varphi_i \|} | \pi_{s-1}(\lambda_i) | \\
&\overset{Lemma\ 3}{\leq} 2 \frac{\prod_{j=1, j\neq i}^m | \lambda_j^{(m)} - \lambda_i |}{\prod_{j=1}^m | \lambda_j^{(m)} |} \gamma_m cond(S_m) \frac{1}{\| P_m \varphi_i \|} \varepsilon_i^{(m)} \frac{\sum_{j=1, j\neq i}^n | \alpha_j \pi_{s-1}(\lambda_j) |}{| \alpha_i \pi_{s-1}(\lambda_i) |} | \pi_{s-1}(\lambda_i) | \\
&= F_{m,i} \frac{\sum_{j=1, j\neq i}^n | \alpha_j \pi_{s-1}(\lambda_j) |}{| \alpha_i |}.
\end{aligned}$$

□

In (2), $F_{m,i}$ is independent of the product polynomials of FOM(m). We assume also for simplicity that $\{F_{m,i}\}_{i=1,2,\cdots,n}$ are uniformly bounded with a reasonably small bound (for example, when m is taken to be

large enough, $\| P_m \varphi_i \|$, $\varepsilon_i^{(m)}$ and γ_m may be all modest scalars). A global property of FOM(m) is interpreted with the following two remarks of Theorem 4.

Remark 1. When $s=1$ we have

$$| p_{m,1}(\lambda_i) | \leq F_{m,i} \frac{\sum_{j=1, j\neq i}^{n} | \alpha_j |}{| \alpha_i |}.$$

If the initial vector r_0 of one FOM cycle is nearly deficient in the i th eigenvector component, i. e., $|\alpha_i| << |\alpha_j|$ for $j \neq i$, then $| p_{m,1}(\lambda_i) |$ may be considerably large. On the other hand, since r_m becomes rich in this component, $| p_{m,2}(\lambda_i) |$ will be correspondingly small in the next FOM cycle. Then $p_{m,2}(z)$ can act as a balance to $p_{m,1}(z)$ in reducing the iteration residual in this direction.

Remark 2. Assuming that r_0 is equally rich in all the eigenvector components, i.e. $|\alpha_1| = |\alpha_2| = \cdots = |\alpha_n|$, then it is obtained from (2) that

$$| \pi_s(\lambda_i) | \leq F_{m,i} \sum_{j=1, j\neq i}^{n} | \pi_{s-1}(\lambda_j) | \quad (i = 1,2,\cdots n),$$

which implies that the modulus of the product polynomial $\pi_s(z)$ on different eigenvalues will be mutually restricted. Consequently, the iteration residual will be simultaneously reduced on the spectrum of A.

A comprehensive discussion can be given for other intermediate cases. In conclusion, with s increasing, $\pi_s(z)$ can strike a balance in residual reduction among all the eigenvector components, in some sense that

$$| \pi_s(\lambda_1) | \approx | \pi_s(\lambda_2) | \approx \cdots \approx | \pi_s(\lambda_n) |, \tag{3}$$

which implies that the FOM(m) cycles can complement one another harmoniously in reducing the iteration residual.

3. Applications of the global property of FOM(m)

The average work per iteration for general restarted Krylov subspace methods such as FOM(m) is proportional to mn; large values of m generally improve convergence but also increase the work per iteration. A considerably cheaper algorithm is polynomial preconditioning coupled with the basic one-step iterative method, namely,

$$r_{km} = [p_m(A)]^k r_0, \; k = 1,2,\cdots, \tag{4}$$

where the polynomial $p_m(z)$ is chosen in some appropriate fashion, satisfying $p_m(0) = 1$. Provide that a good polynomial can be found, this algorithm require only order n work per iteration, independent of m.

Good polynomials can not always be found. For a trivial example, consider the following 2×2 linear system $Ax = b$ with

$$A = \begin{bmatrix} \lambda & \\ & c\lambda \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \; x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $\lambda \neq 0$ and $c >> 1$. We take $m = 1$. The iteration (3) can yield rapid convergence only if $p_1(z) = 1 + \alpha z$ satisfies that $| p_1(\lambda) |$ and $| p_1(c\lambda) |$ both lie well enough below 1. However, when $| p_1(c\lambda) | \leq 1$ is imposed, we have $-2 \leq \alpha c \lambda \leq 0$, which gives that $| p_1(\lambda) | \geq 1 - 2/c$. Inevitably, the iteration (3) will be very slowly convergent for large c, e.g. 100.

The global property of FOM(m) indicates a much more sophisticated procedure to construct good polynomial preconditioners: we may select a set of polynomials to give rapid convergence globally over all s-step cycles rather than locally for only one cycle. In particular, when it is observed FOM(m) converges steadily, a product polynomial of FOM(m) with $s = 2$ can be selected, which works quite well in our limited experiments. In fact, if the balance (3) is attained for FOM(m), then the product polynomial π_s can yield a convergence rate, in an asymptotic sense, as quick as that of FOM(m), resulting in greatly decreased work requirements.

Here is how the coefficients of $p_m(z)$ are calculated explicitly, which follows closely the lines of the

construction of GMRES residual polynomials presented in [8]. Let K_m denote the $n \times m$ matrix of Krylov vectors

$$K_m = (r_0 \quad Ar_0 \quad \cdots \quad A^{m-1} r_0).$$

The Aronldi procedure constructs an $n \times m$ matrix of orthonormal vectors spanning the same space

$$V_m = (v_1 \quad v_2 \quad \cdots \quad v_m),$$

by applying the iterative formula

$$v_{m+1} = h_{m+1,m}^{-1}(Av_m - V_m h_m), \quad h_m = (h_{1m}, \cdots, h_{mm})^T. \tag{5}$$

Since the columns of V_m and K_m span the same space for each *m*, we have that

$$V_m = K_m C_m$$

for some upper-triangular matrix

$$C_m = \begin{pmatrix} c_{11} & \cdots & c_{1m} \\ & \ddots & \vdots \\ & & c_{mm} \end{pmatrix}.$$

This matrix is not formed during the FOM iteration, but to find $p_m(z)$ explicitly we will need it. The appropriate formula comes from (5):

$$\begin{pmatrix} c_{1,m+1} \\ \vdots \\ c_{m+1,m+1} \end{pmatrix} = h_{m+1,m}^{-1} \begin{pmatrix} 0 \\ c_{1,m} \\ \vdots \\ c_{m,m} \end{pmatrix} - h_{m+1,m}^{-1} \begin{pmatrix} C_m h_m \\ 0 \end{pmatrix}. \tag{6}$$

By inserting the calculation (6) in the FOM iteration, we generate the elements of C_m column by column as the iteration proceeds.

Having solved a Hessenberg linear system at step *m*, FOM produces an iterate x_m of the form

$$x_m = x_0 + V_m y$$

for some vector *y* of dimension *m*. Writing $C_m y = (\alpha_1, \cdots, \alpha_m)^T$, it follows from $V_m y = K_m C_m y$ that

$$\begin{aligned} r_m &= b - Ax_m = r_0 - AV_m y \\ &= r_0 - AK_m C_m y = (I - \alpha_1 A - \alpha_2 A^2 - \cdots - \alpha_m A^m) r_0 \equiv p_m(A) r_0. \end{aligned}$$

This gives us the coefficients of $p_m(z)$.

Polynomial preconditioning may also be applied to the restarted Arnoldi method for solving eigenvalue problems. The preconditioning takes the form of a polynomial applied to the starting vector that is constructed to damp unwanted components from the eigenvectors expansion. The resulting algorithm takes the form:

Algorithm 3: Restarted Arnoldi with polynomial preconditioning

(1) Start: Choose an initial vector and normalize for v_1, a number of Arnoldi steps *m.*

(2) Iterate:

(2.1) Perform *m* steps of the Arnoldi iteration starting with v_1, resulting in

$$V_m^T A V_m = H_m.$$

(2.2) Compute eigenpairs $\{\lambda_i^{(m)}, g_i\}$ of H_m and stop if desired eigenpairs are well approximated.

(2.3) Construct a polynomial p to damp unwanted components.

(2.4) $v_1 \leftarrow p(A)v_1$; $v_1 \leftarrow v_1 / \| v_1 \|$ and go back to (2.1).

The construction of the polynomial at step (2.3) may be guided by a priori information about the spectrum of A or solely by information gleaned from H_m. A typical scheme is to sort the spectrum of H_m

into two disjoint sets Ω_w and Ω_u, with $\sigma(H_m) = \Omega_w \cup \Omega_u$. The Ritz values in the set Ω_w are regarded as approximations to the "wanted" eigenvalues of A. The polynomial p is then constructed to be small on Ω_u in comparison with its modulus on Ω_w.

In the above problem the discrete set Ω_u may be replaced by a domain containing it but excluding Ω_w. Let C_u be an open convex domain that contain Ω_u with $\Omega_w \cup C_u = \phi$. Chebyshev polynomials are appropriate when C_u is restricted to be an ellipse. This was proposed by Saad when he adapted the Manteuffel idea to eigenvalue calculations. The resulting algorithm is called an Arnoldi-Chebyshev iteration, see [11].

Computing an optimal ellipse and then carrying out the Chebyshev iteration are quite complicated for practical calculations. Theorem 4 indicates a much simpler alternative of getting an desired polynomial that is small on Ω_u in comparison with its modulus on Ω_w. If FOM is applied to the $m \times m$ linear system $H_m(y - y_o) = r_o$ and r_0 is taken to be a linear combination of the eigenvectors of H_m associated with Ω_u, then by Remark 1 of Theorem 4, after k ($k < m$) steps the resulting FOM residual polynomial will be considerably large on Ω_w, in comparison with its modulus on Ω_u. The following is an outline of this algorithm.

Algorithm 4: Arnoldi-FOM

(1) Start: Choose an initial vector and normalize for v_1, a number of Arnoldi steps m and a number of FOM steps k.

(2) Iterate:

(2.1) Perform m steps of the Arnoldi iteration starting with v_1. Compute the m eigenvalues of the resulting Hessenberg matrix H_m. Select Ω_u and compute the associated eigenvalues. If satisfied stop.

(2.2) Perform k steps of FOM to the linear system $H_m(y - y_o) = r_o$ with $y_0 = 0$ and r_0 being a linear combination of the eigenvectors of H_m associated with Ω_u. Construct the FOM residual polynomial p_k.

(2.3) $v_1 \leftarrow p_k(A)v_1$; $v_1 \leftarrow v_1 / \| v_1 \|$ and go back to (2.1).

4. Numerical experiments

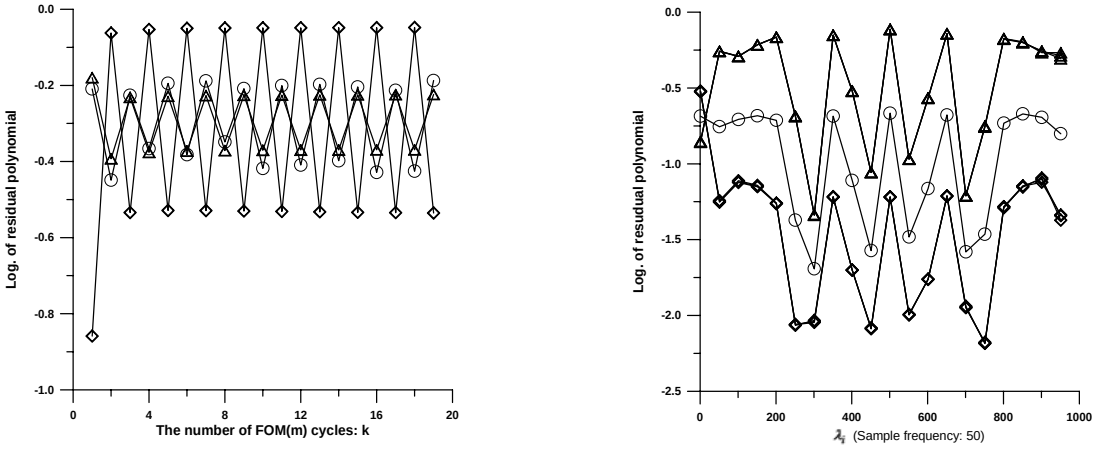
We choose a rather simple example in order to have all the information available to understand the convergence behavior of the restarted Arnoldi method. The matrix is of the form $A = SBS^{-1}$ with $S, B \in R^{1000\times1000}$ selected to be

$$S = \begin{bmatrix} 1 & 0.1 & & \\ & 1 & \ddots & \\ & & \ddots & 0.1 \\ & & & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 1000 \end{bmatrix}. \tag{7}$$

First the global property of FOM(m) is numerically illustrated. Then the efficiency of product FOM polynomials used as preconditioning in solving linear systems is examined. Finally the ability of FOM polynomials used as preconditioning in the restarted Arnoldi method for eigenvalue problems is tested. The right-hand side for the first two tests is chosen as $b = (1,1,\cdots,1)^T$ and the initial guess x_0 is taken to be zero.

Plotting $\log_{10} | p_{m,k}(\lambda_i) |$ against k for some randomly chosen λ_i, Fig. 1(a) shows the harmonious collaboration of FOM(20) cycles applied to this problem. We see that $p_{20,k}(z)$ always has large reduction when k is even and small reduction when k is odd for some eigenvalues, e.g., $\lambda = 500$. Meanwhile, the opposite holds for all the other eigenvalues, e.g., $\lambda = 1; 1000$. This well illustrates Remark 1 of Theorem 4. In more details, in Fig. 1(b) it can be seen that the FOM(m) polynomials of all the odd/even cycles are almost the same (the first few polynomials are not included because they may be strongly influenced by initial vectors). On the other hand, they are clearly distinguished for odd cycles and even cycles. The line of $\sqrt{\pi_2(z)} = \sqrt{p_{m,k}(z)p_{m,k+1}(z)}$ is more flat than those of $p_{m,k}(z)$ and $p_{m,k+1}(z)$. Although the condition (3) are not ideally satisfied here, there exists a balance between local maxima of $\pi_2(z)$. This observation

can be taken as a more meaningful description of the global property of FOM(m), since these maxima dominate the convergence of iteration.



(a) (b)

Fig. 1 (a) The lines marked with △, ◇ and ○ are correspond to $\lambda = 1; 500; 1000$ respectively; (b) The lines marked with △, ◇ and ○ are correspond to $\{p_{m,k}(z)\}_{k=3,5,7}$, $\{p_{m,k}(z)\}_{k=4,6,8}$ and $\sqrt{\pi_2(z)} = \sqrt{p_{m,5}(z)p_{m,6}(z)}$ respectively.

Next, polynomial preconditioning using FOM product polynomials (referred to as FOM P.P.P. in Table 2, $\pi_2(z)$ is constructed from the fifth and sixth FOM(20) cycles in order to remove the influence of initial vectors) is applied to the problem (7). The test was performed on an Intel Pentium IV/1.70G using Visual FORTRAN Professional Edition 5.0A. The algorithms selected for comparison are FOM(20), GMRES(20) and hybrid GMRES of [8] with a switchover step number 20. The convergence tolerance is $\varepsilon = 10^{-10}$. Table 2 gives the run time for each of the algorithms.

Table 2. CPU seconds to convergence for each of the algorithms

Algorithm	FOM P.P.P.	FOM(20)	GMRES(20)	Hybrid GMRES
CPU seconds	32.78	54.46	53.95	81.12

For this problem FOM(20) and GMRES(20) do about equally well. The hybrid GMRES algorithm, which can also be regarded as a polynomial preconditioning and is closely related to our new algorithm, lags far behind. In fact, assuming that δ is the average number of nonzero elements per row of A, the work per step for polynomial preconditioning will be $1+\delta$ vector operations (see [8, Section 6]). Therefore, low sparsity of A will hinder the convergence rate of polynomial preconditionings, just as observed in this example. However, we see that the new algorithm outperforms FOM(20) and GMRES(20) clearly. This shows the appealing property of FOM product polynomials when used as a preconditioning.

Finally we discuss FOM used as a preconditioning in the restarted Arnoldi method for eigenvalue problems. For clarity only the step (2.2) of the Arnoldi-FOM algorithm is considered in detail. After performing 40 steps of the Arnoldi algorithm to the problem (7) with an initial vector v_1 normalized from $(1,1,\cdots,1)^T$, the eigenpairs of the resulting Hensenberg matrix H_m are computed. The set Ω_w is selected as 10 eigenvalues of largest real parts: $\{\overline{\lambda}_1, \overline{\lambda}_2, \cdots, \overline{\lambda}_{10}\}$, and then $\Omega_u = \{\overline{\lambda}_{11}, \overline{\lambda}_{12}, \cdots, \overline{\lambda}_{40}\}$. Let $\{g_{11}, g_{12}, \cdots, g_{40}\}$ be the normalized eigenvectors of H_m associated with Ω_u. We perform k steps of FOM to the linear system

$$H_m(y - y_o) = r_o,$$

where $y_0 = 0$ and r_0 is simply taken as $\sum_{i=11}^{40} g_i$. The factor

$$\tau_k = \max_{\bar{\lambda} \in \Omega_u} | p_k(\bar{\lambda}) | \Big/ \min_{\bar{\lambda} \in \Omega_w} | p_k(\bar{\lambda}) |$$

is then computed to examine the efficiency of the FOM preconditioning - the less τ_k is, the more efficiently the preconditioning works. Some of its values are given in Table 3.

Table 3. Some values of τ_k for different k

k	8	12	16	20	24
τ_k	0.6677	0.4325	0.2109	0.093	0.026

5. Conclusions

In this paper we have presented a study on the restarted Arnoldi method for large unsymmetric matrix problems. In particular, with some simplification assumptions a global property of FOM(m) has been described. Although for realistic problems FOM(m) may behave much more intricately, we believe our findings can help understand the Arnoldi procedure significantly more than before.

Based on the study of FOM(m), two polynomial preconditioning techniques using FOM residual polynomials are introduced. However, there remain many problems to be investigated. For example, it is observed that FOM product polynomial preconditioning lacks robustness to difficult problems. In this case other preconditionings may be used first, or, it can be accelerated, for example, by applying a Krylov subspace method to the preconditioned system. We hope a fuller understanding of these techniques will come with further analysis, experiments, and algorithmic development.

Interesting papers related to our study include [3,6,7,13], which have focused on improving the information generated by GMRES and FOM at restart time by including the spectral information of A, in the form of eigenvalue or eigenvector approximations. In particular, in [7] a restarted FOM algorithm augmented with eigenvectors is proposed, in which some Ritz vectors are added to the Krylov subspace at the time of restarting in order to deflate the corresponding eigenvalues and thus improve the convergence. This process is worthy to be analyzed again in more detail with our theoretical results.

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