ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 1, No. 1, 2006, *pp*. 21-28

Fairing of Parametric Quintic Splines

Yuanjun Wang [1,2] +

[1] Shanghai Institute of Sports, Shanghai 200438, China

[2] School of Mathematical Science, Fudan University, Shanghai 200433, China

(*Received July 31, 2005, accepted Oct. 01, 2005*)

Abstract. In this paper, we gave a fairing algorithm for parametric quintic spline curves. This new algorithm gives a clear modification to the bad point's position, tangent vector, and second tangent vector. We also proved that this algorithm is of energy optimization.

Keywords: curve fairing, quintic parametric spline, energy, optimization.

1. Introduction

In the field of curve fairing, there are two main problems. One is how to quantify the fairness of a curve, which tends to be a divergent problem. Nevertheless, according to the principle of energy optimization [1], a beautiful design is often simple in shape. In this paper, we take the energy criteria as a tool to judge the curve's fairness. Another problem is the methods of fairing. Usually, fairing methods can be classed as interactive and automatical ones. In general, automatical methods can obtain a faster approximation to an optimum than interactive ones [2]. Automatic fairing algorithm can also be further classified as local or global fairing methods. Global ones have better global fairing result, but it is time-consuming because of its large calculations. Local ones only have local fairing effect but they are quickly to perform. In parametric spline fairing, Kjellander proposed a local fairing method to uniform parametric cubic splines [3]. Poliakoff extended the mehod to non-uniform parametric cubic splines [4], and later an automatic fairing algorithm was presented [5]. In our other paper [6], we improved Poliakoff's fairing algorithm to cubic parametric splines by changing the bad points' position and its tangent vector.

In this paper, we present a local fairing algorithm for quintic parametric splines. The bad control points are modified using a local energy optimization. The paper is organized as follows: in section 2 we gave the fairing algorithm to parametric quintic spline curves; in section 3 we presented a simple example; in section 4, we gave the conclusion.

2. The fairing algorithm to parametric quintic spline curves

We suppose that for some integer n, a parametric quintic spline curve passes through data points $\left\{P_i\left(t_i, \boldsymbol{r}_i, \boldsymbol{r}_i', \boldsymbol{r}_i''\right)\right\}_{i=1}^{n}$. Then it is assumed that the curve needs to be faired at one bad data point P_k for some $k(1<k<n)$. Suppose a point on the i^{th} segment of the quintic spline can be rewritten as

$$\boldsymbol{r}(t)=\sum_{j=0}^{5}\boldsymbol{a}_{i,j}\left(t-t_i\right)^j, \quad t_i \le t \le t_{i+1}, \quad t_{i+1}-t_i=\Delta_i, \quad i=1,..,n-1. \tag{1}$$

If the end conditions are $\boldsymbol{r}\left(t_i\right)=\boldsymbol{r}_i, \boldsymbol{r}'\left(t_i\right)=\boldsymbol{r}_i', \boldsymbol{r}''(t_i)=\boldsymbol{r}_i'', \boldsymbol{r}\left(t_{i+1}\right)=\boldsymbol{r}_{i+1}, \boldsymbol{r}'\left(t_{i+1}\right)=\boldsymbol{r}_{i+1}', \boldsymbol{r}''(t_{i+1})=\boldsymbol{r}_{i+1}''$. the $\boldsymbol{a}_{i,j}$ are given by

+ *E-mail address*: 022018041@fudan.edu.cn, wyj803@yahoo.com.cn

$$\boldsymbol{a}_{i,0}=\boldsymbol{r}_i,\boldsymbol{a}_{i,1}=\boldsymbol{r}_i',\boldsymbol{a}_{i,2}=\frac{\boldsymbol{r}_i''}{2},\boldsymbol{a}_{i,3}=\frac{1}{\Delta_i^3}(10\boldsymbol{b}_1-4\boldsymbol{b}_2\Delta_i+\frac{1}{2}\boldsymbol{b}_3\Delta_i^2),$$
$$\boldsymbol{a}_{i,4}=\frac{1}{\Delta_i^4}(-15\boldsymbol{b}_1+7\boldsymbol{b}_2\Delta_i-\boldsymbol{b}_3\Delta_i^2),\boldsymbol{a}_{i,5}=\frac{1}{\Delta_i^5}(6\boldsymbol{b}_1-3\boldsymbol{b}_2\Delta_i+\frac{1}{2}\boldsymbol{b}_3\Delta_i^2). \quad (2)$$

Where $\boldsymbol{b}_1=\boldsymbol{r}_{i+1}-\boldsymbol{r}_i-\boldsymbol{r}_i'\Delta_i-\frac{\boldsymbol{r}_i''}{2}\Delta_i^2,\boldsymbol{b}_2=\boldsymbol{r}_{i+1}'-\boldsymbol{r}_i'-\boldsymbol{r}_i''\Delta_i,\boldsymbol{b}_3=\boldsymbol{r}_{i+1}''-\boldsymbol{r}''$.

Now we use $W=C\int_{t_1}^{t_n}\left(\boldsymbol{r}''(t)\right)^2dt$ to appropriate the curve's internal strain energy. And according to our fairing criteria, the smaller the curve's energy is, the more fair the curve is, to fair the curve by modifying the bad point P_i is equal to minimize the curve's strain energy. We'll give our fairing algorithm by minimizing the curve's strain energy in the following.

Since $\boldsymbol{r}''(t)=2\boldsymbol{a}_{i,2}+6\boldsymbol{a}_{i,3}(t-t_i)+12\boldsymbol{a}_{i,4}(t-t_i)^2+20\boldsymbol{a}_{i,5}(t-t_i)^3$, so the i^{th} section curve's strain energy W_i is $W_i=C\int_{t_i}^{t_{i+1}}\left(\boldsymbol{r}''(t)\right)^2dt=C\int_{t_i}^{t_{i+1}}\left[2\boldsymbol{a}_{i,2}+6\boldsymbol{a}_{i,3}(t-t_i)+12\boldsymbol{a}_{i,4}(t-t_i)^2+20\boldsymbol{a}_{i,5}(t-t_i)^3\right]^2dt$.

From equal (2), we can see $\boldsymbol{a}_{i,2},\boldsymbol{a}_{i,3},\boldsymbol{a}_{i,4},\boldsymbol{a}_{i,5}$ including $\boldsymbol{r}_i,\boldsymbol{r}_i',\boldsymbol{r}_i'',\boldsymbol{r}_{i+1},\boldsymbol{r}_{i+1}',\boldsymbol{r}_{i+1}''$, not including $\boldsymbol{r}_k,\boldsymbol{r}_k',\boldsymbol{r}_k''(k\neq i,i+1)$. In other words, only the control points P_i,P_{i+1} can affect the i^{th} section curve's strain energy W_i. By contrast, if we change the control point P_k only, the $k-1^{th},k^{th}$ section curve's strain energy W_{k-1},W_k will be affected and all other $W_i\,(i\neq k-1,k)$ will be unchanged. So $W_i(new)=W_i\,(i\neq k-1,k)$. Let ΔW denote the variation of the curve's energy $W(new)-W$. Then

$$\Delta W=W(new)-W=\sum_{i=1}^{n-1}\left(W_i(new)-W_i\right)=W_{k-1}(new)-W_{k-1}+W_k(new)-W_k.$$

For conference, we give the following signal:

$\left\{P_i\left(t_i,\boldsymbol{r}_i,\boldsymbol{r}_i',\boldsymbol{r}_i''\right)\right\}_{i=1}^n$ ~ *the primitive data points*

$\left\{P_i\left(t_i,\boldsymbol{r}(new)_i,\boldsymbol{r}_i'(new),\boldsymbol{r}_i''(new)\right)\right\}_{i=1}^n$ ~ *the modified data points*

$\boldsymbol{r}(t)$ ~ *the primitive interpolating spline curve*

$\boldsymbol{r}(new)(t)$ ~ *the new interpolating spline curve.*

$\Delta P_i=(\boldsymbol{\delta}_{r_i},\boldsymbol{\eta}_{r_i'},\boldsymbol{\xi}_{r_i''})$ ~ *the change of* P_i

$\boldsymbol{\delta}_{r_i}=(\delta_{x_i},\delta_{y_i},\delta_{z_i})$ ~ *the change of the position of* P_i

$\boldsymbol{\eta}_{r_i'}=(\eta_{x_i'},\eta_{y_i'},\eta_{z_i'})$ ~ *the change of the first derivative of* P_i

$\boldsymbol{\xi}_{r_i''}=(\xi_{x_i''},\xi_{y_i''},\xi_{z_i''})$ ~ *the change of the second derivative of* P_i

Now we give our algorithm: $P_k(new)=P_k+\Delta P_k$, that is

$$\begin{cases}\boldsymbol{r}_k(new)=\boldsymbol{r}_k+\boldsymbol{\delta}_{\boldsymbol{r}_k}\\ \boldsymbol{r}_k'(new)=\boldsymbol{r}_k'+\boldsymbol{\eta}_{\boldsymbol{r}_k'}\\ \boldsymbol{r}_k''(new)=\boldsymbol{r}_k''+\boldsymbol{\xi}_{r_k''}\end{cases} \quad (3)$$

Where

$$\begin{cases} \boldsymbol{\delta}_{\mathbf{r}_k} = \dfrac{2}{15}\dfrac{\Delta_k^3\Delta_{k-1}^3}{\left(\Delta_k+\Delta_{k-1}\right)^3}\left[-C_1+\dfrac{25}{16}\dfrac{\Delta_{k-1}-\Delta_k}{\Delta_k\Delta_{k-1}}C_2+\dfrac{5}{\Delta_k\Delta_{k-1}}C_3\right] \\ \boldsymbol{\eta}_{\mathbf{r}'_k} = \dfrac{5}{24}\dfrac{\left(\Delta_k-\Delta_{k-1}\right)\Delta_k^2\Delta_{k-1}^2}{\left(\Delta_k+\Delta_{k-1}\right)^3}\left[-C_1+\dfrac{2(\Delta_{k-1}-\Delta_k)}{\Delta_k\Delta_{k-1}}C_2+\dfrac{5}{\Delta_k\Delta_{k-1}}C_3\right] \\ \boldsymbol{\xi}_{\mathbf{r}''_k} = \dfrac{2}{3}\dfrac{\Delta_k^2\Delta_{k-1}^2}{\left(\Delta_k+\Delta_{k-1}\right)^3}\left[C_1+\dfrac{25}{16}\dfrac{(\Delta_k-\Delta_{k-1})}{\Delta_k\Delta_{k-1}}C_2-\dfrac{5}{16}\dfrac{\left(7\Delta_k^2+7\Delta_{k-1}^2+18\Delta_k\Delta_{k-1}\right)}{\Delta_k^2\Delta_{k-1}^2}C_3\right] \end{cases} \tag{4}$$

And

$$\begin{cases} C_1 = 12\left(\boldsymbol{a}_{k,3}-\boldsymbol{a}_{k-1,3}\right)+24\left(\boldsymbol{a}_{k,4}\Delta_k-\boldsymbol{a}_{k-1,4}\Delta_{k-1}\right)+\dfrac{240}{7}\left(\boldsymbol{a}_{k,5}\Delta_k^2-\boldsymbol{a}_{k-1,5}\Delta_{k-1}^2\right) \\ C_2 = -4(\boldsymbol{a}_{k,2}-\boldsymbol{a}_{k-1,2})+12\boldsymbol{a}_{k-1,3}\Delta_{k-1}+\dfrac{24}{5}\boldsymbol{a}_{k,4}\Delta_k^2+\dfrac{96}{5}\boldsymbol{a}_{k-1,4}\Delta_{k-1}^2+\dfrac{64}{7}\boldsymbol{a}_{k,5}\Delta_k^3+\dfrac{176}{7}\boldsymbol{a}_{k-1,5}\Delta_{k-1}^3 \\ C_3 = \dfrac{2}{5}\left(\boldsymbol{a}_{k,4}\Delta_k^3+\boldsymbol{a}_{k-1,4}\Delta_{k-1}^3\right)+\dfrac{6}{7}\boldsymbol{a}_{k,5}\Delta_k^4+\dfrac{8}{7}\boldsymbol{a}_{k-1,5}\Delta_{k-1}^4 \end{cases}$$

Lemma. Changing P_k to $P_k\left(new\right)$, all other points unchanged, the change of internal strain energy is

$$W^*-W = CI\left(\boldsymbol{\delta}_{\mathbf{r}_k},\boldsymbol{\eta}_{\mathbf{r}'_k},\boldsymbol{\xi}_{\mathbf{r}''_k}\right) \tag{5}$$

$$\begin{aligned} I\left(\boldsymbol{\delta}_{\mathbf{r}_k},\boldsymbol{\eta}_{\mathbf{r}'_k},\boldsymbol{\xi}_{\mathbf{r}''_k}\right) = &\left[12\left(\boldsymbol{a}_{k,3}-\boldsymbol{a}_{k-1,3}\right)+24\left(\boldsymbol{a}_{k,4}\Delta_k-\boldsymbol{a}_{k-1,4}\Delta_{k-1}\right)+\frac{240}{7}\left(\boldsymbol{a}_{k,5}\Delta_k^2-\boldsymbol{a}_{k-1,5}\Delta_{k-1}^2\right)\right]\cdot\boldsymbol{\delta}_{\mathbf{r}_k} \\ &+\left[-4\left(\boldsymbol{a}_{k,2}-\boldsymbol{a}_{k-1,2}\right)+12\boldsymbol{a}_{k-1,3}\Delta_{k-1}+\frac{24}{5}\boldsymbol{a}_{k,4}\Delta_k^2+\frac{96}{5}\boldsymbol{a}_{k-1,4}\Delta_{k-1}^2+\frac{64}{7}\boldsymbol{a}_{k,5}\Delta_k^3+\frac{176}{7}\boldsymbol{a}_{k-1,5}\Delta_k^3\right]\cdot\boldsymbol{\eta}_{\mathbf{r}'_k} \\ &+\left[\frac{2}{5}\left(\boldsymbol{a}_{k,4}\Delta_k^3+\boldsymbol{a}_{k-1,4}\Delta_{k-1}^3\right)+\frac{6}{7}\boldsymbol{a}_{k,5}\Delta_k^4+\frac{8}{7}\boldsymbol{a}_{k-1,5}\Delta_{k-1}^4\right].\boldsymbol{\xi}_{\mathbf{r}''_k} \\ &+\left[\frac{120}{7}\left(\frac{1}{\Delta_k^3}+\frac{1}{\Delta_{k-1}^3}\right)\boldsymbol{\delta}_{\mathbf{r}_k}^2+\frac{192}{35}\left(\frac{1}{\Delta_k}+\frac{1}{\Delta_{k-1}}\right)\boldsymbol{\eta}_{\mathbf{r}'_k}^2+\frac{3}{35}\left(\Delta_{k-1}+\Delta_k\right)\boldsymbol{\xi}_{\mathbf{r}''_k}^2\right. \\ &\left.+\frac{120}{7}\left(\frac{1}{\Delta_k^2}-\frac{1}{\Delta_{k-1}^2}\right)\boldsymbol{\delta}_{\mathbf{r}_k}\cdot\boldsymbol{\eta}_{\mathbf{r}'_k}+\frac{6}{7}\left(\frac{1}{\Delta_k}+\frac{1}{\Delta_{k-1}}\right)\boldsymbol{\delta}_{\mathbf{r}_k}.\boldsymbol{\xi}_{\mathbf{r}''_k}\right] \end{aligned}$$

***Proof*:** For $1\le i\le n-1$, $W_i(new) = C\int_{t_i}^{t_{i+1}}\left(\boldsymbol{r}''\left(new\right)\left(t\right)\right)^2 dt$, noticing $\boldsymbol{r}\left(new\right)\left(t\right)=\sum_{j=0}^{5}\boldsymbol{a}_{i,j}\left(t-t_i\right)^j$,

and

$$\boldsymbol{r}''\left(new\right)\left(t\right) = 2\boldsymbol{a}_{i,2}+6\boldsymbol{a}_{i,3}\left(t-t_i\right)+12\boldsymbol{a}_{i,4}\left(t-t_i\right)^2+20\boldsymbol{a}_{i,5}\left(t-t_i\right)^3.$$

So

$$W_i(new) = C\int_{t_i}^{t_{i+1}}\left[2\boldsymbol{a}_{i,2}+6\boldsymbol{a}_{i,3}\left(t-t_i\right)+12\boldsymbol{a}_{i,4}\left(t-t_i\right)^2+20\boldsymbol{a}_{i,5}\left(t-t_i\right)^3\right]^2 dt.$$

As recited before, $\Delta W = W\left(new\right)-W = \sum_{i=1}^{n-1}\left(W_i\left(new\right)-W_i\right) = W_{k-1}\left(new\right)-W_{k-1}+W_k\left(new\right)-W_k$. Since

$$W_k = C\int_{t_k}^{t_{k+1}}\left[\left(x''\left(t\right)\right)^2+\left(y''\left(t\right)\right)^2+\left(z''\left(t\right)\right)^2\right]dt = W_{k,x}+W_{k,y}+W_{k,z},$$

where $W_{k,x} = C\int_{t_k}^{t_{k+1}} \left(x''(t)\right)^2 dt$, the same to $W_{k,y}$, $W_{k,z}$. Now suppose that P_k is changed by ΔP_k, taking the x-component of $\boldsymbol{a}_{i,j}$ (for clarity, we also denote it with $a_{i,j}$). We can see that each $a_{i,j}$ is a function of x_k, x'_k, x''_k, but not of y_k, y'_k, y''_k ; z_k, z'_k, z''_k. So the change of x-component $W_{k,x}(new) - W_{k,x}$ can be expressed using Taylor formula.

$$\begin{aligned}
&W_{k,x}(new)\left(x_k(new), x'_k(new), x''_k(new)\right) - W_{k,x}\left(x_k, x'_k\right) \\
&= W_{k,x}(new)\left(x_k + \delta_{x_k}, x'_k + \eta_{x'_k}, x''_k + \xi_{x''_k}\right) - W_{k,x}\left(x_k, x'_k, x''\right) \\
&= (W_{k,x}(new))'_{x_k}\delta_{x_k} + (W_{k,x}(new))'_{x'_k}\eta_{x'_k} + (W_{k,x}(new))'_{x''_k}\xi_{x''_k} + \frac{1}{2!}\Big[(W_{k,x}(new))''_{x_k x_k}\delta_{x_k}^2 + (W_{k,x}(new))''_{x'_k x'_k}\eta_{x'_k}^2 + (W_{k,x}(new))''_{x''_k x''_k}\xi_{x''_k}^2 \\
&+ 2(W_{k,x}(new))''_{x_k x'_k}\delta_{x_k}\eta_{x'_k} + 2(W_{k,x}(new))''_{x_k x''_k}\delta_{x_k}\xi_{x''_k} + 2(W_{k,x}(new))''_{x'_k x''_k}\eta_{x'_k}\xi_{x''_k}\Big] + R_3\left(\delta_{x_k}, \eta_{x'_k}, \xi_{x''_k}\right)
\end{aligned}$$

By calculating, we can get

$$\begin{aligned}
&(a_{k,2})'_{x_k} = (a_{k,2})'_{x'_k} = 0, (a_{k,2})'_{x''_k} = \frac{1}{2} \\
&(a_{k,3})'_{x_k} = -\frac{10}{\Delta_k^3}, (a_{k,3})'_{x'_k} = -\frac{6}{\Delta_k^2}, (a_{k,3})'_{x''_k} = -\frac{3}{2\Delta_k} \\
&(a_{k,4})'_{x_k} = \frac{15}{\Delta_k^4}, (a_{k,4})'_{x'_k} = \frac{8}{\Delta_k^3}, (a_{k,4})'_{x''_k} = \frac{3}{2\Delta_k^2} \\
&(a_{k,5})'_{x_k} = -\frac{6}{\Delta_k^5}, (a_{k,5})'_{x'_k} = -\frac{3}{\Delta_k^4}, (a_{k,3})'_{x''_k} = -\frac{1}{2\Delta_k^3}
\end{aligned}$$

Then

$$(W_{k,x}(new))'_{x_k} = C\left[12a_{k,3} + 24a_{k,4}\Delta_k + \frac{240}{7}a_{k,5}\Delta_k^2\right]$$

$$(W_{k,x}(new))'_{x'_k} = C\left[-4a_{k,2} + \frac{24}{5}a_{k,4}\Delta_k^2 + \frac{64}{7}a_{k,5}\Delta_k^3\right]$$

$$(W_{k,x}(new))'_{x''_k} = C\left[\frac{2}{5}a_{k,4}\Delta_k^3 + \frac{6}{7}a_{k,5}\Delta_k^4\right]$$

$$(W_{k,x}(new))''_{x_k x_k} = C\frac{240}{7\Delta_k^3}, \quad (W_{k,x}(new))''_{x'_k x'_k} = C\frac{384}{35\Delta_k}, \quad (W_{k,x}(new))''_{x''_k x''_k} = C\frac{6\Delta_k}{35}$$

$$(W_{k,x}(new))''_{x'_k x_k} = (W_{k,x}(new))''_{x_k x'_k} = C\frac{120}{7\Delta_k^2}$$

$$(W_{k,x}(new))''_{x_k x''_k} = (W_{k,x}(new))''_{x''_k x_k} = C\frac{6}{7\Delta_k}$$

$$(W_{k,x}(new))''_{x'_k x''_k} = (W_{k,x}(new))''_{x''_k x'_k} = C\frac{22}{35}$$

All other higher derivatives of $W_{k,x}(new)$ to x_k, x'_k, x''_k are zero, so $R_3\left(\delta_{x_k}, \eta_{x'_k}, \xi_{x''_k}\right)$ is zero. Therefore,

$$W_{k,x}(new)\left(x_k(new),x_k'(new),x_k''(new)\right)-W_{k,x}\left(x_k,x_k',x_k''\right)$$
$$=C\Big\{\left[12a_{k,3}+24a_{k,4}\Delta_k+\frac{240}{7}a_{k,5}\Delta_k^2\right]\delta_{x_k}+\left[-4a_{k,2}+\frac{24}{5}a_{k,4}\Delta_k^2+\frac{64}{7}a_{k,5}\Delta_k^3\right]\eta_{x_k'}$$
$$+\left[\frac{2}{5}a_{k,4}\Delta_k^3+\frac{6}{7}a_{k,5}\Delta_k^4\right]\xi_{x_k''}+\frac{1}{2!}\Big[\frac{240}{7\Delta_k^3}\delta_{x_k}^2+\frac{384}{35\Delta_k}\eta_{x_k'}^2+\frac{6}{35}\Delta_k\xi_{x_k''}^2+\frac{240}{7\Delta_k^2}\delta_{x_k}\eta_{x_k'}+$$
$$\frac{12}{7\Delta_k}\delta_{x_k}\xi_{x_k''}+\frac{44}{35}\eta_{x_k'}\xi_{x_k''}\Big]\Big\}$$

and similarly we can get

$$W_{k-1,x}(new)\left(x_{k-1}(new),x_{k-1}'(new),x_{k-1}''(new)\right)-W_{k-1,x}\left(x_{k-1},x_{k-1}',x_{k-1}''\right)$$
$$=C\Big\{\left[-12a_{k-1,3}-24a_{k-1,4}\Delta_{k-1}-\frac{240}{7}a_{k-1,5}\Delta_{k-1}^2\right]\delta_{x_k}+\Big[4a_{k-1,2}+12a_{k-1,3}\Delta_{k-1}+\frac{96}{5}a_{k-1,4}\Delta_{k-1}^2$$
$$+\frac{176}{7}a_{k-1,5}\Delta_{k-1}^3\Big]\eta_{x_k'}+\left[\frac{2}{5}a_{k-1,4}\Delta_{k-1}^3+\frac{8}{7}a_{k-1,5}\Delta_{k-1}^4\right]\xi_{x_k''}+\frac{1}{2!}\Big[\frac{240}{7\Delta_{k-1}^3}\delta_{x_k}^2+\frac{384}{35\Delta_{k-1}}\eta_{x_k'}^2+\frac{6}{35}\Delta_{k-1}\xi_{x_k''}^2$$
$$-\frac{240}{7\Delta_{k-1}^2}\delta_{x_k}\eta_{x_k'}+\frac{12}{7\Delta_{k-1}}\delta_{x_k}\xi_{x_k''}-\frac{44}{35}\eta_{x_k'}\xi_{x_k''}\Big]\Big\}$$

add the above two equals, and replace x with $\boldsymbol{r}$, we get $W(new)-W=CI\left(\boldsymbol{\delta}_{r_k},\boldsymbol{\eta}_{r_k'},\boldsymbol{\xi}_{r_k''}\right)$, where

$$I\left(\boldsymbol{\delta}_{r_k},\boldsymbol{\eta}_{r_k'},\boldsymbol{\xi}_{r_k''}\right)=\left[12\left(\boldsymbol{a}_{k,3}-\boldsymbol{a}_{k-1,3}\right)+24\left(\boldsymbol{a}_{k,4}\Delta_k-\boldsymbol{a}_{k-1,4}\Delta_{k-1}\right)+\frac{240}{7}\left(\boldsymbol{a}_{k,5}\Delta_k^2-\boldsymbol{a}_{k-1,5}\Delta_{k-1}^2\right)\right]\cdot\boldsymbol{\delta}_{r_k}$$
$$+\left[-4\left(\boldsymbol{a}_{k,2}-\boldsymbol{a}_{k-1,2}\right)+12\boldsymbol{a}_{k-1,3}\Delta_{k-1}+\frac{24}{5}\boldsymbol{a}_{k,4}\Delta_k^2+\frac{96}{5}\boldsymbol{a}_{k-1,4}\Delta_{k-1}^2+\frac{64}{7}\boldsymbol{a}_{k,5}\Delta_k^3+\frac{176}{7}\boldsymbol{a}_{k-1,5}\Delta_k^3\right]\cdot\boldsymbol{\eta}_{r_k'}$$
$$+\left[\frac{2}{5}\left(\boldsymbol{a}_{k,4}\Delta_k^3+\boldsymbol{a}_{k-1,4}\Delta_{k-1}^3\right)+\frac{6}{7}\boldsymbol{a}_{k,5}\Delta_k^4+\frac{8}{7}\boldsymbol{a}_{k-1,5}\Delta_{k-1}^4\right].\boldsymbol{\xi}_{r_k''}$$
$$+\Big[\frac{120}{7}\left(\frac{1}{\Delta_k^3}+\frac{1}{\Delta_{k-1}^3}\right)\boldsymbol{\delta}_{r_k}^2+\frac{192}{35}\left(\frac{1}{\Delta_k}+\frac{1}{\Delta_{k-1}}\right)\boldsymbol{\eta}_{r_k'}^2+\frac{3}{35}\left(\Delta_{k-1}+\Delta_k\right)\boldsymbol{\xi}_{r_k''}^2$$
$$+\frac{120}{7}\left(\frac{1}{\Delta_k^2}-\frac{1}{\Delta_{k-1}^2}\right)\boldsymbol{\delta}_{r_k}\cdot\boldsymbol{\eta}_{r_k'}+\frac{6}{7}\left(\frac{1}{\Delta_k}+\frac{1}{\Delta_{k-1}}\right)\boldsymbol{\delta}_{r_k}.\boldsymbol{\xi}_{r_k''}\Big]$$

Theorem. Assume a parametric quintic spline curve passing through points $\left\{P_i\left(t_i,\boldsymbol{r}_i,\boldsymbol{r}_i',\boldsymbol{r}_i''\right)\right\}_{i=1}^n$, $\Delta_i=t_{i+1}-t_i>0,i=1,\dots,n-1$. We also assume that only one bad point $P_k(1<k<n)$ requires to be modified, and all other points are good. Changing P_k to $P_k(new)$, the new curve interpolating $\left\{P_i(new)\left(t_i,\boldsymbol{r}_i(new),\boldsymbol{r}_i'(new),\boldsymbol{r}''(new)_i\right)\right\}_{i=1}^n$ is of energy optimization.

***Proof*:** From the **Lemma**, the change of the internal strain energy is $W(new)-W=CI\left(\boldsymbol{\delta}_{r_k},\boldsymbol{\eta}_{r_k'},\boldsymbol{\xi}_{r_k''}\right)$.

Let

$$\begin{cases} \left(I\left(\delta_{r_k},\eta_{r'_k},\xi_{r''_k}\right)\right)'_{\delta_{r_k}} = 0 \\ \left(I\left(\delta_{r_k},\eta_{r'_k},\xi_{r''_k}\right)\right)'_{\eta_{r'_k}} = 0 \\ \left(I\left(\delta_{r_k},\eta_{r'_k},\xi_{r''_k}\right)\right)'_{\xi_{r''_k}} = 0 \end{cases}$$

that is

$$\begin{cases} C_1 + \frac{240}{7}\left(\frac{1}{\Delta_k^3}+\frac{1}{\Delta_{k-1}^3}\right)\delta_{r_k} + \frac{120}{7}\left(\frac{1}{\Delta_k^2}-\frac{1}{\Delta_{k-1}^2}\right)\eta_{r'_k} + \frac{6}{7}\left(\frac{1}{\Delta_k}+\frac{1}{\Delta_{k-1}}\right)\xi_{r''_k} = 0 \\ C_2 + \frac{120}{7}\left(\frac{1}{\Delta_k^2}-\frac{1}{\Delta_{k-1}^2}\right)\delta_{r_k} + \frac{384}{35}\left(\frac{1}{\Delta_k}+\frac{1}{\Delta_{k-1}}\right)\eta_{r'_k} = 0 \\ C_3 + \frac{6}{7}\left(\frac{1}{\Delta_k}+\frac{1}{\Delta_{k-1}}\right)\delta_{r_k} + \frac{6}{35}\left(\Delta_{k-1}+\Delta_k\right)\xi_{r''_k} = 0 \end{cases}$$

so

$$\begin{cases} \delta_{r_k} = \frac{2}{15}\frac{\Delta_k^3\Delta_{k-1}^3}{\left(\Delta_k+\Delta_{k-1}\right)^3}\left[-C_1 + \frac{25}{16}\frac{\Delta_{k-1}-\Delta_k}{\Delta_k\Delta_{k-1}}C_2 + \frac{5}{\Delta_k\Delta_{k-1}}C_3\right] \\ \eta_{r'_k} = \frac{5}{24}\frac{\left(\Delta_k-\Delta_{k-1}\right)\Delta_k^2\Delta_{k-1}^2}{\left(\Delta_k+\Delta_{k-1}\right)^3}\left[-C_1 + \frac{2(\Delta_{k-1}-\Delta_k)}{\Delta_k\Delta_{k-1}}C_2 + \frac{5}{\Delta_k\Delta_{k-1}}C_3\right] \\ \xi_{r''_k} = \frac{2}{3}\frac{\Delta_k^2\Delta_{k-1}^2}{\left(\Delta_k+\Delta_{k-1}\right)^3}\left[C_1 + \frac{25}{16}\frac{(\Delta_k-\Delta_{k-1})}{\Delta_k\Delta_{k-1}}C_2 - \frac{5}{16}\frac{\left(7\Delta_k^2+7\Delta_{k-1}^2+18\Delta_k\Delta_{k-1}\right)}{\Delta_k^2\Delta_{k-1}^2}C_3\right] \end{cases}$$

Also because the Hession matrix of I_x

$$H\left(I_x\right) = \begin{pmatrix} \left(I_x\right)''_{\delta_{x_k}\delta_{x_k}} & \left(I_x\right)''_{\delta_{x_k}\eta_{x'_k}} & \left(I_x\right)''_{\delta_{x_k}\xi_{x''_k}} \\ \left(I_x\right)''_{\eta_{x'_k}\delta_{x_k}} & \left(I_x\right)''_{\eta_{x'_k}\eta_{x'_k}} & \left(I_x\right)''_{\eta_{x'_k}\xi_{x''_k}} \\ \left(I_x\right)''_{\xi_{x''_k}\delta_{x_k}} & \left(I_x\right)''_{\xi_{x''_k}\eta_{x'_k}} & \left(I_x\right)''_{\xi_{x''_k}\xi_{x''_k}} \end{pmatrix} = \begin{pmatrix} \frac{240}{7}\left(\frac{1}{\Delta_k^3}+\frac{1}{\Delta_{k-1}^3}\right) & \frac{120}{7}\left(\frac{1}{\Delta_k^2}-\frac{1}{\Delta_{k-1}^2}\right) & \frac{6}{7}\left(\frac{1}{\Delta_k}+\frac{1}{\Delta_{k-1}}\right) \\ \frac{120}{7}\left(\frac{1}{\Delta_k^2}-\frac{1}{\Delta_{k-1}^2}\right) & \frac{384}{35}\left(\frac{1}{\Delta_k}+\frac{1}{\Delta_{k-1}}\right) & 0 \\ \frac{6}{7}\left(\frac{1}{\Delta_k}+\frac{1}{\Delta_{k-1}}\right) & 0 & \frac{6}{35}\left(\Delta_{k-1}+\Delta_k\right) \end{pmatrix}$$

is positive. So $\left(\delta_{x_k},\eta_{x'_k},\xi_{x''_k}\right)$ is the only minimum value of $W_x\left(new\right)-W_x$. Then $\left(\delta_{r_k},\eta_{r'_k},\xi_{r''_k}\right)$ is the only minimum value of $W\left(new\right)-W$. Therefor, we say the new curve is of energy optimization.

Corollary. Especially, to uniform quintic spline curves, with all $\Delta_i \equiv 1, (i=1,2,...,n-1)$, the fairing algorithm is $P_k(new) = P_k + \Delta P_k$, $\Delta P_k = \left(\delta_{r_k},\eta_{r'_k},\xi_{r''_k}\right)$. Where $\delta_{r_k},\eta_{r'_k},\xi_{r''_k}$ are expressed as

$$\begin{cases} \delta_{\boldsymbol{r}_k} = -\frac{1}{5}\left(\boldsymbol{a}_{k,3} - \boldsymbol{a}_{k-1,3}\right) + \frac{13}{30}\boldsymbol{a}_{k,4} - \frac{11}{30}\boldsymbol{a}_{k-1,4} + \frac{9}{14}\boldsymbol{a}_{k,5} - \frac{10}{21}\boldsymbol{a}_{k-1,5} \\ \eta_{\boldsymbol{r}'_k} = \frac{35}{192}(\boldsymbol{a}_{k,2} - \boldsymbol{a}_{k-1,2}) - \frac{35}{64}\boldsymbol{a}_{k-1,3} - \frac{7}{32}\boldsymbol{a}_{k,4} - \frac{7}{8}\boldsymbol{a}_{k-1,4} - \frac{5}{12}\boldsymbol{a}_{k,5} - \frac{55}{48}\boldsymbol{a}_{k-1,5} \\ \xi_{\boldsymbol{r}''_k} = \left(\boldsymbol{a}_{k,3} - \boldsymbol{a}_{k-1,3}\right) + \frac{2}{3}\boldsymbol{a}_{k,4} - \frac{10}{3}\boldsymbol{a}_{k-1,4} - \frac{20}{3}\boldsymbol{a}_{k-1,5} \end{cases}$$

3. Examples

Numerous of fairing examples showed that this new method is valid. In this section, we gave a simple example to illustrate this validness.Fig.1 is the primitive interpolating curve. Fig.2 is the curvature figure corresponding to Fig.1. It is easy to judge from Fig.2 that Fig.1 is not fair. Fig.3 is the figure interpolating data points faired by our method. Fig.4 is the curvature figure corresponding to Fig.3. Clearly, our new fairing method is very valid.

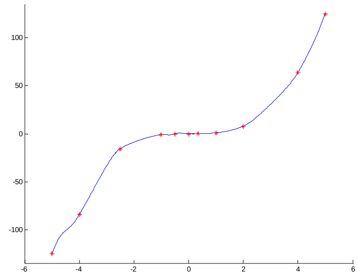

Fig. 1

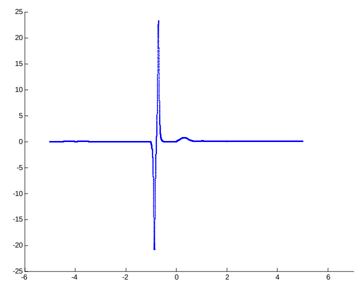

Fig. 2

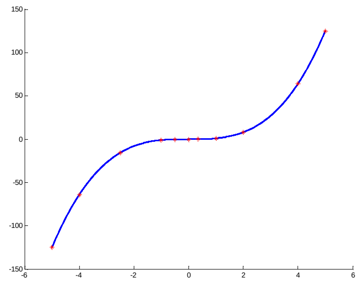

Fig. 3

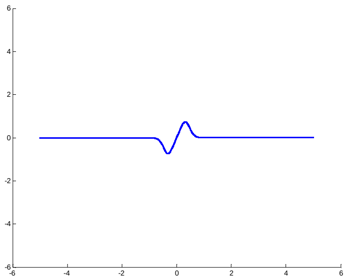

Fig. 4

4. Conclusion

In the fairing of parametric spline, Kjellander and Poliakoff composed their fairing methods to cubic spline curves. In this paper, based on energy criterion, we give a new fairing algorithm to quintic spline curves. The new fairing algorithm not only changes the bad point's position but also changes it's tangent vector, second tangent vector. We also proved the new algorithm is of energy optimization. Numerous examples show that this algorithm is valid.

In future, we'll extend this algorithm to other spline curves. What's more, as the application of surface is more broadly than curve, we'll devote ourselves to the fairing algorithm of the surface.

5. Acknowledgements

Thanks very much to my advisor Mr.Yuan Cao, Associate Professor of Fudan University.

6. References

[1] Xuefu Wang, Fuhua Cheng and Brian A Barsky, Energy and B-Spline Interproximation, CAD, 29(1996)7, 485-496

[2] N. Sapidis and G. Farin, Automatic Fairing Algorithm for B-Spline Curves, CAD, 22(1990)2, 121-129

[3] J.A.P. Kjellander, Smoothing of Cubic Parametric Spline, CAD, 15(1983)3, 175-179

[4] J. F. Poliakoff, An Improved Algorithm for Automatic Fairing of Non-uniform Parametric Cubic Splines, CAD, 26(1996)1, 59-66

[5] J.F. Poliakoff, et al., An Automated Curve Fairing Algorithm for Cubic B-spline Curve", Journal of Computational and Applied Mathematics, 102(1999), 73~85

[6] Yuanjun Wang, Yuan Cao, Energy Optimization Fairing Algorithm Of Non-uniform Parametric Cubic Splines, CAD&CG, 9(2005).