# Asymptotic Iterative Approximation of Intellectualized Periodic Interpolating Spline and its Application [*]

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**Abstract.** This paper presents an efficient and sufficient algorithm to approximate general interpolating Spline curve which goes through the given set of design-data points by using asymptotic iterative B-spline curves. Firstly, the given set is considered as the control points of a B-spline, to create the initial approximate curve, then the iterative grade is built based on the error between the initial approximate curve and given set, it is used to generate an iterative function sequence to approximate the interpolating function of the cover. New algorithms and approaches are successfully employed to create realistic and flexible geometry models for the centerline of yarns in woven fabrics. The algorithm described on this paper will be generalized to wide range for general 2D and 3D geometry and computer graphics with regard to curves and surfaces, especially for complex geometrical models on CAD and CAM.

**Keywords:** Geometric reasoning; Mathematical Model; Computer Graphics, Spline Function, Computational Geometry, Woven Fabrics.

## 1. Introduction

The spline function arises from the concept of spline, a drafting tool. This flexible strip is flexed to form a smooth curve by naturally passing through a set of design-data points which is suitable to describe the geometrical curves. This mechanical/physical spline follows the principle of minimum energy with continuity requirement across the various curve sections [1]. It has been generalized to represent both 3D curve and surface and their comprehensive application.

The methods to create spline curves can be categorized to interpolating splines and approximation splines [1, 2]. When continual polynomial piece are fitted so that the curve passes through each design-data point, the resulting curve is said to interpolate the set of design-data points, the relevant splines are defined as interpolate spline. On the other hand, when the continual polynomial pieces are fitted to the general control-point pass without necessarily passing through any control point the resulting curve is said to approximate the set of control points, the relevant splines are defined as approximation splines. Generally, in order to find out the curve of interpolating spline which passes through each design-data point of interpolating spline, we have to solve the high order linear system [1]. Apparently, it is not suitable for geometrical modelling of complex curves. For the approximation splines including B-spline, etc, although it has been used in CAD and some other research fields [3, 4, 5], a key technical problem to find the approximation splines which pass through the set of designed coordinate positions must be solved in order that it can be successfully applied to complex curves and most application fields. One of possible approaches is to find out control points of Spline according to given design-data points, it must be the job of high complicacy and also is mostly to find out solution from the ill-conditional system. Thus it is not suitable for complex geometrical curve and computer graphics. The novel approach is expected to be described.

This paper presents an efficient and sufficient algorithm to approximate general interpolating Spline curve which goes through the set of design-data points by using asymptotic iterative B-spline curves. New algorithms and approaches are successfully employed to create realistic and flexible geometry models for the centerline of yarns in woven fabrics in this paper. The approach must be generalized to wide range for

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general 2D and 3D geometry and computer graphics with regard to curves and surfaces, especially for complex geometrical models on CAD and CAM.

## 2. Description of the Periodic Interpolating B-splines

The principle of the approach is demonstrated firstly by the curve in 3D space based on parabola B-spline, which can then be further generalized to surface in 3D space and to create based on different styles of splines for both curves and surfaces.

Let the super-vector below

$$
\boldsymbol{h} = \left( \vec{h}_1 \quad \vec{h}_2 \quad \cdots \quad \vec{h}_i \quad \cdots \quad \vec{h}_N \right) = \begin{pmatrix} h_1^x & h_2^x & \cdots & h_i^x & \cdots & h_N^x \\ h_1^y & h_2^y & \cdots & h_i^y & \cdots & h_N^y \\ h_1^z & h_2^z & \cdots & h_i^z & \cdots & h_N^z \end{pmatrix}.
$$

be the set of design-data points in 3D space. The aim of interpolation B-spline is to create a spline curve $\vec{s}(t)$ in 3D space what goes through the set above. Especially for periodic curve, two extra points, $\vec{h}_{1-1}$ and $\vec{h}_{N+1}$ are need. For this reason, an extended super-vector $\boldsymbol{h}^*$ of weft vector $\boldsymbol{h}$ is created which is shown below.

$$
\boldsymbol{h}^* = \left( \vec{h}_0 \quad | \quad \vec{h}_1 \quad \vec{h}_2 \quad \cdots \quad \vec{h}_i \quad \cdots \quad \vec{h}_N \quad | \quad \vec{h}_{N+1} \right),
$$

where $\vec{h}_{N+1} = \vec{h}_1$ and $\vec{h}_0 = \vec{h}_N$.

The algorithm of the approach of periodic interpolating B-splines can now be summarized in the following steps with respect to Fig. 1.
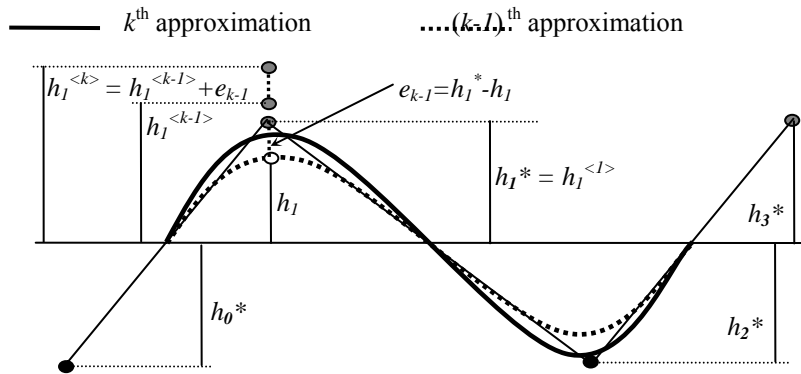


Fig. 1. $k^{th}$ iterative of interpolating B-spline.

**Step 1**: Using $\boldsymbol{h}^*$ as the initial set of control points marked as $\boldsymbol{h}^{<1>}$ ( $k = 1$ ), find out the B-spline $\vec{s}^{<1>}(t)$ by using the parabola B-spline which is described by formula as following

$$
\vec{s}^{<1>}(t) = \vec{s}_i^{<1>}(t) = \frac{1}{2} \left( \vec{h}_{i-1}^{<1>} \quad \vec{h}_i^{<1>} \quad \vec{h}_{i+1}^{<1>} \right) \begin{pmatrix} 1 & -2 & 1 \\ 1 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} \qquad \text{for } i^{th} \text{ piece,} \qquad (1)
$$

$$
0 \le t \le 1, \ i = 1, 2, \ldots, N
$$

where

$$
\vec{s}_i(t) = \begin{pmatrix} x_i(t) \\ y_i(t) \\ z_i(t) \end{pmatrix} \text{ and } \left( \vec{h}_{i-1}^{<k>} \quad \vec{h}_i^{<k>} \quad \vec{h}_{i+1}^{<k>} \right) = \begin{pmatrix} h_{i-1}^{x,<k>} & h_i^{x,<k>} & h_{i+1}^{x,<k>} \\ h_{i-1}^{y,<k>} & h_i^{y,<k>} & h_{i+1}^{y,<k>} \\ h_{i-1}^{z,<k>} & h_i^{z,<k>} & h_{i+1}^{z,<k>} \end{pmatrix}. \qquad (2)
$$

Use

$$
\boldsymbol{h}^{*<1>} = \left( \vec{h}_0^{<1>} \quad | \quad \vec{h}_1^{<1>} \quad \vec{h}_2^{<1>} \quad \cdots \quad \vec{h}_i^{<1>} \quad \cdots \quad \vec{h}_N^{<1>} \quad | \quad \vec{h}_{N+1}^{<1>} \right),
$$

to record the super-vector of values of B-spline with respect to control points $\boldsymbol{h}^{<1>}$. For equidistant nodes,

$$\vec{h}_i^{<1>} = \vec{s}_i^{<1>}(\frac{1}{2}) \quad (i = 1, 2, ..., N), \quad \vec{h}_0^{<1>} = \vec{h}_N^{<1>} \text{ and } \vec{h}_{N+1}^{<1>} = \vec{h}_1^{<1>}$$

**Step 1+**: Calculate the error vector $\boldsymbol{e}^{<1>} = \boldsymbol{h}^* - \boldsymbol{h}^{*<1>}$ to find out the error.

**Step 2**: Get a new set of control points $\boldsymbol{h}^{<2>} = \boldsymbol{h}^{<1>} + \boldsymbol{e}^{<1>}$ and find out the B-spline $\vec{s}^{<2>}(t)$ by formula as following

$$\vec{s}^{<2>}(t) = \vec{s}_i^{<2>}(t) = \frac{1}{2}\begin{pmatrix} \vec{h}_{i-1}^{<2>} & \vec{h}_i^{<2>} & \vec{h}_{i+1}^{<2>} \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} \quad \text{for } i^{\text{th}} \text{ piece,} \tag{3}$$

where $\vec{s}_i(t)$ and $\begin{pmatrix} \vec{h}_{i-1}^{<2>} & \vec{h}_i^{<2>} & \vec{h}_{i+1}^{<2>} \end{pmatrix}$ are defined by formula (2).

Use

$$\boldsymbol{h}^{*<2>} = \begin{pmatrix} \vec{h}_0^{<2>} & | & \vec{h}_1^{<2>} & \vec{h}_2^{<2>} & \cdots & \vec{h}_i^{<2>} & \cdots & \vec{h}_N^{<2>} & | & \vec{h}_{N+1}^{<2>} \end{pmatrix},$$

to record the super-vector of values of B-spline with respect to control points $\boldsymbol{h}^{<2>}$. For equidistant nodes,

$$\vec{h}_i^{<2>} = \vec{s}_i^{<2>}(\frac{1}{2}) \quad (i = 1, 2, ..., N), \quad \vec{h}_0^{<2>} = \vec{h}_N^{<2>} \text{ and } \vec{h}_{N+1}^{<2>} = \vec{h}_1^{<2>}$$

**Step 2+**: Calculate the second error vector $\boldsymbol{e}^{<2>} = \boldsymbol{h}^* - \boldsymbol{h}^{*<2>}$ to find out the error.

……

**Step k**: Get a new set of control points $\boldsymbol{h}^{<k>} = \boldsymbol{h}^{<k-1>} + \boldsymbol{e}^{<k-1>}$ and find out the B-spline $\vec{s}^{<k>}(t)$ by formula as following

$$\vec{s}^{<k>}(t) = \vec{s}_i^{<k>}(t) = \frac{1}{2}\begin{pmatrix} \vec{h}_{i-1}^{<k>} & \vec{h}_i^{<k>} & \vec{h}_{i+1}^{<k>} \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} \quad \text{for } i^{\text{th}} \text{ piece,} \tag{3}$$

where $\vec{s}_i(t)$ and $\begin{pmatrix} \vec{h}_{i-1}^{<k>} & \vec{h}_i^{<k>} & \vec{h}_{i+1}^{<k>} \end{pmatrix}$ are defined by formula (2).

Use

$$\boldsymbol{h}^{*<k>} = \begin{pmatrix} \vec{h}_0^{<k>} & | & \vec{h}_1^{<k>} & \vec{h}_2^{<k>} & \cdots & \vec{h}_i^{<k>} & \cdots & \vec{h}_N^{<k>} & | & \vec{h}_{N+1}^{<k>} \end{pmatrix}, \tag{4}$$

to record the super-vector of values of B-spline with respect to control points $\boldsymbol{h}^{<k>}$. For equidistant nodes,

$$\vec{h}_i^{<k>} = \vec{s}_i^{<k>}(\frac{1}{2}) \quad (i = 1, 2, ..., N), \quad \vec{h}_0^{<k>} = \vec{h}_N^{<k>} \text{ and } \vec{h}_{N+1}^{<k>} = \vec{h}_1^{<k>}. \tag{5}$$

**Step k+**: Calculate the $k^{\text{th}}$ error vector $\boldsymbol{e}^{<k>} = \boldsymbol{h}^* - \boldsymbol{h}^{*<k>}$ to find out the error.

……

In so doing two approximate super-vector sequences can be obtained by (3), (4) and (5), which are

$$\boldsymbol{h}^{*<1>}, \boldsymbol{h}^{*<2>}, ..., \boldsymbol{h}^{*<k>}, \ldots$$

and

$$\boldsymbol{e}^{<1>}, \boldsymbol{e}^{<2>}, ..., \boldsymbol{e}^{<k>}, \ldots$$

and the interpolating B-spline sequence $\vec{s}^{<k>}(t)$ is obtained. It might be approximate to $\vec{s}(t)$ which go through the design-data points.

## 3. The Convergence of the Periodic Interpolating B-splines

For the super-vector sequences above, we can show the theorems below.

**Theorem 1.** For any $i$, the error sequence $h_i^x - h_i^{x,<k>}$ is always keep same sign for all $k$, and the absolute value of $h_i^x - h_i^{x,<k>}$ is a decreasing sequence with regard to $k$. The $h_i^y - h_i^{y,<k>}$ and $h_i^z - h_i^{z,<k>}$ also have

same character, respectively.

**Theorem 2.** The error sequence $e^{<1>}$ is a "degressive super-vector sequence", that means

$$\|e^{<1>}\| \geq \|e^{<2>}\| \geq \ldots \geq \| e^{<k>}\| \geq \ldots$$

**Theorem 3.** The sequence of super-vector $h^{*<k>}$ ( k = 1, 2, … ) is converged at $h^*$, that also can be shown as

$$\lim_{k\to\infty} \|e^{<k>}\| = \lim_{k\to\infty} \| h^* - h^{*<k>} \| = \lim_{k\to\infty} ( h^* - h^{*<k>})^T (h^* - h^{*<k>}) = 0$$

and

$$\lim_{k\to\infty} h^{*<k>} = h^*.$$

The theorems indicated that $h^{*<k>}$ and $e^{<k>}$ converge at $h^*$ and $0$ respectively. That means the B-spline sequence $\vec{s}^{<k>}(t)$ is approximate to special piecewise curve $\vec{s}(t)$ which go through the design-data points. The curve can be called as "interpolating spline".

## 4. Numerical Example

A numerical example to implement the interpolate B-spline approximation is given below. Let the design-data points as

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 10 & -10 & 10 & -10 & 10 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and extended it to

$$h^* = \left( \begin{array}{c|cccccc|c} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 10 & 10 & -10 & 10 & -10 & 10 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

It is used to be the first set of control points of B-spline. Then created the periodic interpolating B-spline sequence $\vec{s}^{<k>}(t)$ by using the approach mentioned on the section 2 above to find out the final Spline curve which goes through the given design-data points. The Fig. 2 described the approximate curve. And the table 1 listed the numerical sequence on the approximate process. The table shows us the approach provided a quick and efficient method to implement the interpolating Spline without solving any system of implicit function equation.
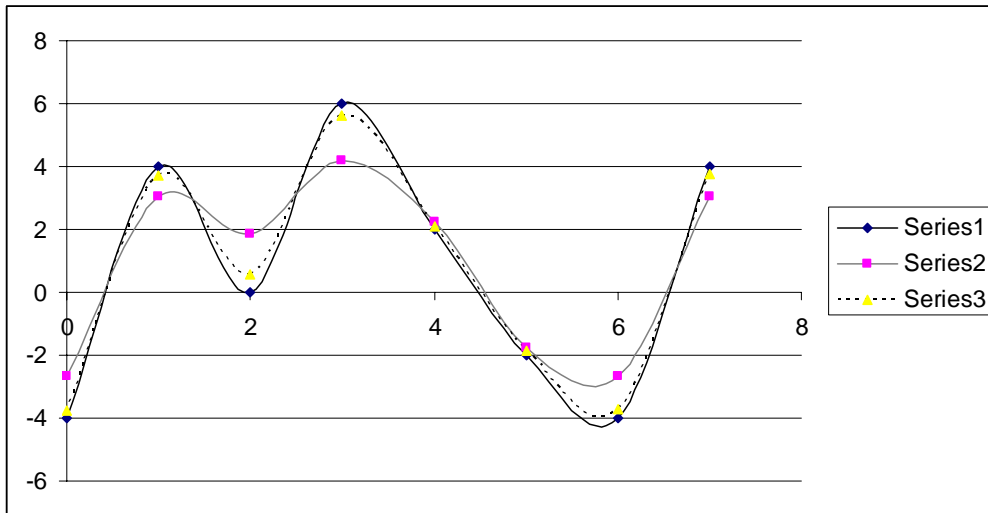


Fig. 2. a numerical example of the periodic interpolated approximate B-spline.

Table1. The data of the approximate process.

|  | $i=0$ | $i=1$ | $i=2$ | $I=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ |
|---|---|---|---|---|---|---|---|---|
| $k=1$ | -4.00 | 4.00 | 0.00 | 6.00 | 2.00 | -2.00 | -4.00 | 4.00 |
| $k=2$ | -2.65 | 3.05 | 1.85 | 4.21 | 2.25 | -1.75 | -2.65 | 3.05 |
| $k=3$ | -3.71 | 3.71 | 0.55 | 5.61 | 2.08 | -1.85 | -3.71 | 3.71 |
| $k=4$ | -3.91 | 3.90 | 0.15 | 5.52 | 2.04 | -1.91 | -3.91 | 3.90 |
| $k=5$ | -3.97 | 3.97 | 0.06 | 5.81 | 2.01 | -1.97 | -3.97 | 3.97 |
| $k=6$ | -3.99 | 3.99 | 0.02 | 5.93 | 2.01 | -1.99 | -3.99 | 3.99 |
| $k=7$ | -4.00 | 4.00 | 0.01 | 5.98 | 2.01 | -1.99 | -4.00 | 4.00 |
| $k=8$ | -4.00 | 4.00 | 0.00 | 5.99 | 2.00 | -2.00 | -4.00 | 4.00 |
| Design-data Points | -4.00 | 4.00 | 0.00 | 6.00 | 2.00 | -2.00 | -4.00 | 4.00 |

## 5. Application on Modelling of Woven Fabrics

Woven fabric is the special composite structure which consists of warp and weft yarns by different pattern. The yarns are fundamental structural elements of fabric. The very complex geometrical structures of fabric are mainly determined by the centreline configurations of their constituent yarns, which vary quite widely and are depended on the weave patterns, the tightness of construction, the processing parameters and external interaction. So the most important thing for creating the geometrical model of woven fabric is to get good flexible representation of yarns. Especially, the software package is need to generate arbitrary curves automatically those go through the given set of principal points of yarn in woven fabric based on mechanical calculation.

Firstly, once the fabric pattern and setting are designed, the structure of initial vector matrix $H_0$ of deflection of yarns can be obtained below.

$$H_0 = \begin{pmatrix} \vec{h}_{R_f,1} & \vec{h}_{R_f,2} & \cdots & \vec{h}_{R_f,i} & \cdots & \vec{h}_{R_f,R_w} \\ \vdots & \vdots & & \vdots & & \vdots \\ \vec{h}_{j,1} & \vec{h}_{j,2} & \cdots & \vec{h}_{j,i} & \cdots & \vec{h}_{j,R_w} \\ \vdots & \vdots & & \vdots & & \vdots \\ \vec{h}_{2,1} & \vec{h}_{2,2} & \cdots & \vec{h}_{2,i} & \cdots & \vec{h}_{2,R_w} \\ \vec{h}_{1,1} & \vec{h}_{1,2} & \cdots & \vec{h}_{1,i} & \cdots & \vec{h}_{1,R_w} \end{pmatrix}. \tag{6}$$

Then we can calculate the warp yarn deflection or weft yarn deflection based on the relationship between deflection and thicknesses of yarn which can be shown by formula $h_w + h_f = d_w + d_f$. This with the fabric pattern helps us to give initial yarn deflections to obtain the value of matrix (6) for creating the yarn spline curves.

As mentioned on the section above, parabola spline function can be considered as an appropriate choice for most of yarns in fabric. A subprogram has been developed to implement the above approach by using the parabola B-spline with the matrix (6). The relevant piecewise parametrical expression $h_j(t)$ of the fundament spline formula in $k^{\text{th}}$ approximation for $j^{\text{th}}$ weft yarn is shown below:

$$h_j(t) = \vec{h}_{j,i}(t) = \frac{1}{2}\begin{pmatrix} \vec{h}_{j,i-1}^{<k>} & \vec{h}_{j,i}^{<k>} & \vec{h}_{j,i+1}^{<k>} \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} \qquad \text{for } i^{\text{th}} \text{ piece,} \tag{7}$$

$$0 \leq t \leq 1, \ i = 1, 2, \ldots, R_w$$

where

$$\vec{h}_{j,i}(t) = \begin{pmatrix} x_{j,i}(t) \\ y_{j,i}(t) \\ z_{j,i}(t) \end{pmatrix}, \quad \begin{pmatrix} \vec{h}_{j,i-1}^{<k>} & \vec{h}_{j,i}^{<k>} & \vec{h}_{j,i+1}^{<k>} \end{pmatrix} = \begin{pmatrix} h_{j,i-1}^{x,<k>} & h_{j,i}^{x,<k>} & h_{j,i+1}^{x,<k>} \\ h_{j,i-1}^{y,<k>} & h_{j,i}^{y,<k>} & h_{j,i+1}^{y,<k>} \\ h_{j,i-1}^{z,<k>} & h_{j,i}^{z,<k>} & h_{j,i+1}^{z,<k>} \end{pmatrix}$$

and $t$ is a parameter which changes between 0 and 1 to match the starting point and end point of each piecewise Spline of yarn respectively.

The approach above has been implemented by C++. Now we just show tow examples whose created by the software package.

Fig. 3 is a fabric model with 2/1/3/1 twill structure. The cross-sections of warp and weft yarn are elliptical with the long axis being 0.59 mm and short axis being 0.36mm for warp, long axis being 0.90mm and short axis being 0.59 mm for weft. The warp and weft densities in the fabric are 10 ends/cm and 10 picks/cm.
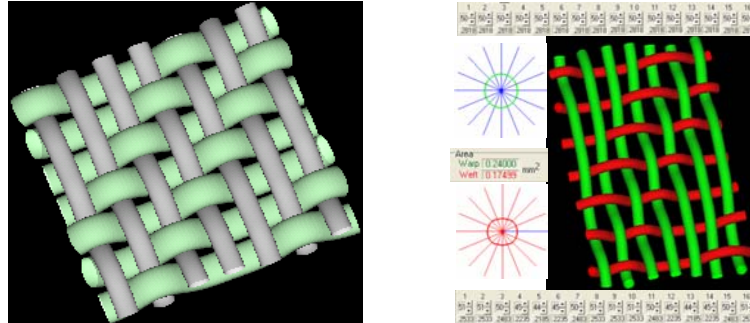


Fig. 3. Graphics of woven fabric with 2/1/3/1 structure based on the approach.
Fig. 4. Graphics of woven fabric with Irregular cross-section.

In Fig. 4, the fabric mode was created as 2/1/3/1 structure with 6picks/cm and 9ends/cm. The warp yarns are drawn by the irregular cross-section with 16 nodes that have same radii. The weft yarns are described also by the irregular cross-section with 16 nodes that have different radii. The areas of cross-sections of warp and weft are 0.240mm$^2$ and 0.175mm$^2$ respectively.

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