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The Preconditioned Simultaneous Displacement Method for Linear Complementarity Problems *

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Abstract. In this paper, a preconditioned simultaneous displacement method is investigated for the large sparse linear complementarity problems. Under suitable restrictions on the involved parameters, convergence results for this method are presented when the system matrix is an H-matrix with positive diagonal elements.

Keywords: Linear complementarity problem, H-matrix, iteration, convergence.

1. Introduction

For a given matrix $A \in R^{n\times n}$ and $b \in R^n$, the linear complementarity problem LCP(A,b) consists of finding a vector which satisfies the conditions

$$x \ge 0, \quad Ax + b \ge 0, \quad x^T(Ax+b) = 0. \tag{1}$$

This problem arises in various scientific computing areas such as the Nash equilibrium point of a bimatrix game, contact problems, the free boundary problem for journal bearings, etc., see [8].

Over the years, many methods for solving the LCP(A,b) have been developed. Most of the methods have their origin in the solution of linear systems and may be classified into two categories, pivoting methods and iterative methods. Iterative methods, which generate an infinite sequence converging to a solution of the problem, are particularly effective for large and sparse problems. Recently, much attention has been paid on the class of iterative methods, which is an extension of the matrix splitting method for solving linear systems. Cottle et al. [8] presented detailed descriptions about these methods. In [8] they studied the convergence of the splitting method.

In this paper, we will study the convergence of the preconditioned simultaneous displacement method for the linear complementarity problems. Under suitable restrictions on the involved parameters, we prove the convergence of the preconditioned simultaneous displacement method for solving the linear complementarity problem when its system matrix is an H-matrix.

In the following, we first present some basic concepts, definitions and some well-known results which shall be used later. Then, in Section 3 and Section 4, we will focus on the preconditioned simultaneous displacement method and present the convergence results for this method when the coefficient matrix is an H-matrix with positive diagonal elements

2. Preliminaries

We shall use the following notations. For $A=(a_{ij}), B=(b_{ij}) \in R^{n\times n}$, we write $A \le B$ if $a_{ij} \le b_{ij}$ holds for all $i, j = 1,2,...,n.$ A is nonnegative if $A \ge 0$, this definition carries immediately over to vectors by identifying them with $n \times 1$ matrices. In particular, we call the vector $x \in R^n$ positive (written $x > 0$) if all its entries are positive. Let the diag(A) $\in R^{n\times n}$ be the diagonal matrix with the same diagonal elements as in $A=(a_{ij}) \in R^{n\times n}$. By $|A| = (|a_{ij}|)$ and $\langle A \rangle$ we denote the absolute value of A and the comparison matrix of A, respectively. the comparison matrix $\langle A \rangle = (\alpha_{ij}) \in R^{n\times n}$ is defined such that

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$$\alpha_{ij} = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases} \quad i = 1, 2, \cdots, n.$$

Note that an H-matrix is nonsingular, and has the properties that $|A^{-1}| \leq \langle A \rangle^{-1}$. Spectral radius of A is denoted by $\rho(A)$.

Definition 1. *A matrix* $A = [a_{ij}] \in R^{n\times n}$ *is called*

(i) *a strictly diagonally dominant (SDD) matrix if*

$$|a_{ii}| > r_i(A) := \sum_{j\neq i} |a_{ij}|, \quad i = 1, 2, \ldots, n$$

(ii) *a* L*-matrix if* $a_{ii} > 0$ *and* $a_{ij} \leq 0, i \neq j; i, j = 1, 2, ..., n,$

(iii) *an* M *-matrix if it is a nonsingular* L*-matrix satisfying* $A^{-1} \geq 0$,

(iv) *an* H *-matrix if* $\langle A \rangle$ *is an* M *-matrix.*

Lemma 1 [7]. *Let A is a nonnegative irreducible matrix, then* $\rho(A)$ *is an eigenvalue and its corresponding eigenvector* $x > 0$.

Lemma 2 [1]. *Let* $A \in R^{n\times n}$ *be an H-matrix with positive diagonal elements. Then the LCP(* A, b *) has a unique solution* $z^* \in R^n$.

3. The preconditioned simultaneous displacement method for LCP(A, b)

Linear complementarity problems can be transformed to equivalent fixed-point system of equations [8]. Thus solving LCP(A, b) is equivalent to finding a solution of the system

$$g(x) = 0, \tag{2}$$

where $g(x) = \min(x, Ax + b)$, "$\min(u, v)$" denotes the componentwise minimum of two vectors *u* and *v*. Note

$$\min(u, v) = \frac{1}{2}(u + v - |u - v|) \quad \forall u, v \in R^n,$$

such that (2) is equivalent to

$$(I + A)x + b = |(I + A)x - b| \tag{3}$$

where I is the identity matrix. Let

$$A = D - L - U \triangleq D - B$$

be the standard splitting of A into diagonal (D), strictly lower (L) and strictly upper (U) triangular matrices, respectively. D is supposed to be nonsingular.

From (3), it follows that

$$\tau(I + D)^{-1}(I + A)x = \tau(I + D)^{-1}[|(I - A)x - b| - b]$$

and let

$$\begin{aligned} \tau(I + D)^{-1}(I + A) &= \tau(I + D)^{-1}[(I + D) - L - U] \\ &= \tau[I - (I + D)^{-1}L - (I + D)^{-1}U] \\ &= I - \omega(I + D)^{-1}(L + U) + \omega^2(I + D)^{-1}L(I + D)^{-1}U \\ &\quad -(1 - \tau)I - (\tau - \omega)(I + D)^{-1}(L + U) \\ &\quad -\omega^2(I + D)^{-1}L(I + D)^{-1}U \\ &= [I - \omega(I + D)^{-1}L][I - \omega(I + D)^{-1}U] \\ &\quad -[(1 - \tau)I + (\tau - \omega)(I + D)^{-1}B + \omega^2(I + D)^{-1}L(I + D)^{-1}U] \end{aligned}$$

where τ and ω are real parameters.

Let us denote

$$M = [I - \omega[(I + D)^{-1}L][I - \omega(I + D)^{-1}U],$$
$$N = (1 - \tau)I + (\tau - \omega)(I + D)^{-1}(L + U) + \omega^2(I + D)^{-1}L(I + D)^{-1}U$$

such that

$$\tau(I + D)^{-1}(I + A) = M - N.$$

The preconditioned simultaneous displacement (PSD) method for LCP(A, b) is defined as follows:

PSD method

Step 1: Choose an initial approximation $x^0 \in R^n$ and set $k = 0$;
Step 2: Calculate

$$x^{k+1} = M^{-1}\{Nx^k + \tau(I+D)^{-1}[|(I-A)x^k - b| - b]\}; \tag{4}$$

Step 3: If $x^{k+1} = x^k$ then stop, otherwise set $k := k+1$ and return to Step 2.

4. Convergence

In this section, we emphatically discuss the convergence of the PSD method when the system matrix A is an H-matrix. We suppose that

$$D = \text{diag}(A), \quad \det(I+D) \neq 0$$

and

$$A = D - B, \quad J = (I+D)^{-1}B,$$

$$\rho_J = \rho(|J|), \quad \sigma_J = 2\rho_J + \max_{1\le i\le n}\frac{|1-a_{ii}|}{|1+a_{ii}|} < 1.$$

Theorem. *Let the parameters τ and ω satisfy*

$$0 \le \omega \le \tau, \quad 0 < \tau < \frac{2}{1+\sigma_J}.$$

Then, for any initial guess $x^0 \in R^n$, the iterative sequence $\{x^k\}$ generated by the PSD method converges to the unique solution x^* of the LCP.

Proof. By Lemma 2, we know that the LCP(A, b) has a unique solution x^*, then

$$x^* = M^{-1}Nx^* + \tau(I+D)^{-1}[|(I-A)x^* - b| - b].$$

and denote the absolute value of the "error vector" by

$$e^k = |x^k - x^*|, \quad k = 0,1,2,\cdots.$$

i.e., the difference between the k th iterate and the exact solution. After some algebra, we have

$$e^{k+1} \le |M^{-1}|[|N| + \tau|(I+D)^{-1}||I-A|]e^k, \tag{5}$$

where

$$\begin{aligned}|M^{-1}| &= |[I-\omega(I+D)^{-1}U]^{-1}[I-\omega(I+D)^{-1}L]^{-1}| \\ &\le |[I-\omega(I+D)^{-1}U]^{-1}||[I-\omega(I+D)^{-1}L]^{-1}|\end{aligned}$$

and

$$\begin{aligned}|N| &= |(1-\tau)I + (\tau-\omega)(I+D)^{-1}B + \omega^2(I+D)^{-1}L(I+D)^{-1}U| \\ &\le |1-\tau|I + (\tau-\omega)|I+D|^{-1}|B| + \omega^2|I+D|^{-1}|L||I+D|^{-1}|U|.\end{aligned}$$

Note that $[I-\omega(I+D)^{-1}U]$ is a strictly diagonally dominant (SDD) matrix, so it is an H-matrix. Thus, we have

$$|[I-\omega(I+D)^{-1}U]^{-1}| \le \langle I-\omega(I+D)^{-1}U\rangle^{-1} \le (I-\omega|I+D|^{-1}|U|)^{-1}$$

and

$$(I-\omega|I+D|^{-1}|U|)^{-1} \ge 0.$$

Similarly,

$$|[I-\omega(I+D)^{-1}L]^{-1}| \le (I-\omega|I+D|^{-1}|L|)^{-1}$$

and

$$(I-\omega|I+D|^{-1}|L|)^{-1} \ge 0.$$

Let us denote

$$\begin{aligned}L_{PSD}(\omega,\tau) = (I-\omega|I+D|^{-1}|U|)^{-1}(I-\omega|I+D|^{-1}|L|)^{-1}\{|1-\tau|I + (\tau-\omega)|I+D|^{-1}|B| \\ +\omega^2|I+D|^{-1}|L||I+D|^{-1}|U| + \tau|I+D|^{-1}|I-A|\}.\end{aligned} \tag{6}$$

By (5) and (6), we have

$$e^{k+1} \le L_{PSD}(\omega,\tau)e^k \tag{7}$$

Let

$$J_\varepsilon = |I+D|^{-1}|B| + \varepsilon ee^T, \quad e = (1,1,\cdots,1) \tag{8}$$

Note that J_ε is a nonnegative irreducible matrix. From Lemma 1, we know that there exists $x_\varepsilon > 0$, such that

$$J_\varepsilon x_\varepsilon = \rho(J_\varepsilon) x_\varepsilon \tag{9}$$

By $\sigma(J) < 1$ we have $\rho(J) < 1$. Hence, when ε is adequate, we can obtain

$$\rho_\varepsilon = \rho(J_\varepsilon) < 1, \quad \sigma_\varepsilon = 2\rho_\varepsilon + \max_{1\le i\le n} \frac{|1-a_{ii}|}{|1+a_{ii}|} < 1 \tag{10}$$

and

$$|1-\tau| + \tau\sigma_\varepsilon < 1 \tag{11}$$

Following (6) and (8), we have

$$\begin{aligned} L_{PSD}(\omega,\tau) &= I + (I-\omega|I+D|^{-1}|U|)^{-1}(I-\omega|I+D|^{-1}|L|)^{-1}\{(|1-\tau|-1)I \\ &\quad +2\tau|I+D|^{-1}|B| + \tau|(I+D)^{-1}||I-D|\} \\ &\le I + (I-\omega|I+D|^{-1}|U|)^{-1}(I-\omega|I+D|^{-1}|L|)^{-1}\{(|1-\tau|-1)I \\ &\quad +2\tau J_\varepsilon + \tau|(I+D)^{-1}||I-D|\} \end{aligned} \tag{12}$$

Using (9)-(12), we can get

$$\begin{aligned} L_{PSD}(\omega,\tau)x_\varepsilon &\le \{I + (I-\omega|I+D|^{-1}|U|)^{-1}(I-\omega|I+D|^{-1}|L|)^{-1}[(|1-\tau|-1)I \\ &\quad +2\tau J_\varepsilon + \tau|(I+D)^{-1}||I-D|]\}x_\varepsilon \\ &\le \{I + (I-\omega|I+D|^{-1}|U|)^{-1}(I-\omega|I+D|^{-1}|L|)^{-1}\cdot[(|1-\tau|-1)+\tau+\tau\sigma_\varepsilon]\}x_\varepsilon \\ &\le x_\varepsilon \end{aligned}$$

So from [7], the inequality

$$\rho(L_{PSD}(\omega,\tau)) \le \sigma_\varepsilon < 1$$

holds and the theorem is proved. □

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