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A Numerical Method for Finding Positive Solution of Dirichlet Problem with a Weight Function

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Abstract. Using a numerical method based on sub-super solution, we will show the existence of positive solution for the problem $-\Delta u = \lambda g(x) f(u(x))$ for $x \in \Omega$, with Dirichlet boundary condition.

Keywords: Stable Solution, Positive Solutions, Sub and Super-solutions.

1. Introduction

In this paper, we consider the existence of positive solution of the semi-linear elliptic boundary value problem

$$\begin{cases} -\Delta u(x) = g(x) f(u(x)) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \tag{1}$$

where Ω is a bounded region in R^n with smooth boundary , $g : \Omega \to R$ is a smooth function. We will assume throughout that f satisfies

(f1) $f : I \to R^+$ is a smooth function where $I = [0, r]$ or $[0, \infty)$, $f(0) = 0$ and $f''(u) < 0$ for all $u \in I$.

We investigate numerically positive solutions. Our numerical method is based on monotone iteration.

We say that u is a positive solution of (1) if u is a classical solution with $u(x) \in I$ for all $x \in \bar{\Omega}$ and $u(x) > 0$ for all $x \in \Omega$.

Our study of (1) is motivated by the fact that the equations arises in population genetics (see [5]) in which case the function g attains both positive and negative values on Ω. In the case when $g \equiv 1$ it is well known that (1) has at most one non-constant positive solution when f satisfies (f1) (see [4]) but may have multiple solution when f is convex (see [1]).

Theorem 1. Suppose f satisfies (f1). If u is a positive non-constant solution of (1) then the smallest eigenvalue of the linearized problem associated with (1), viz,

$$-\Delta\psi - g(x) f'(u(x))\psi = \mu\psi \quad in\Omega, \qquad \psi = 0 \quad on\partial\Omega$$

is positive.

The above theorem (proved in [2]) shows that all non-constant positive solution of (1) are non-degenerate and stable, so we can use the method of sub-super solution. We shall now assume also that (f2) $I = [0,1]$ and $f(1) = 0$, e.g., $f(u) = u(1-u)(\gamma(1-u) + (1-\gamma)u)$ as studied in [5] which also satisfy (f1) provided $\frac{1}{3} < \gamma < \frac{2}{3}$.

We can now investigate the multiplicity of solutions of

$$-\Delta u - g(x) f(u) = 0 in \quad \Omega, \quad u = 0 \quad on \quad \partial\Omega. \tag{2}$$

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Theorem 2. Suppose f satisfies (f1) and (f2).

(i) Suppose that the principal eigenvalues of $-\Delta - g(x)f^{'}(0)$ with Dirichlet boundary conditions is positive, i.e., 0 is a stable solution of (2). Then (2) has no positive solution.

(ii) Suppose that the principal eigenvalues of $-\Delta - g(x)f^{'}(0)$ with Dirichlet boundary conditions is negative. Then (2) has exactly one positive solution.

Theorem (3) also proved in [2] that we give a proof for part *(ii)* for the sake of completeness.

Since f satisfies (f2), $u \equiv 1$ is a super solution of (2) and so the iteration defined by $u_1 \equiv 1$ and

$$-\Delta u_{n+1} + Cu_{n+1} = g(x)f(u_n) + Cu_n \quad in \quad \Omega; \quad u_{n+1} = 0 \;\; on \partial\Omega$$

gives a monotonic decreasing sequence of supersolution of (2) converging to the maximal solution of (2). Thus, if $\overline{u} = u_2$, i.e., the unique solution of

$$-\Delta u + Cu = g(x)f(1) + Cu \quad in \quad \Omega; \quad u = 0 \;\; on \quad \partial\Omega$$

then $\overline{u}$ is a supersolution of (2) which is greater than or equal to the maximal solution of (2). Hence all positive solutions of (2) lie in $[0, \overline{u}]$ and proved is complete.

Finally we have

Theorem 4. [2] Suppose f satisfies the hypotheses of Theorem 2 and g changes sign on Ω. Then the problem $-\Delta u = \lambda g(x) f(u)$ has no positive solution if $0 \le \lambda \le \lambda_1$ and exactly one positive solution when $\lambda > \lambda_1$.

2. Numerical Results

It is well-known that there must always exist a solution for problems such as (2) between a sub-solution $\underline{v}$ and a super-solution $\overline{u}$ such that $\underline{v} \le \overline{u}$ for all $x \in \Omega$ (see [3]).

Consider the boundary value problem

$$\begin{cases} \Delta u(x) + f(x, u(x)) = 0 & on \;\; \Omega \\ u(x) = 0 & on \;\; \partial\Omega. \end{cases} \tag{3}$$

Let $\overline{u}$, $\underline{u} \in C^2(\overline{\Omega})$ satisfy $\overline{u} \ge \underline{u}$ as well as

$$\Delta \overline{u}(x) + f(x, \overline{u}(x)) \le 0 \quad on \Omega \quad \overline{u} \ge 0$$

$$\Delta \underline{v}(x) + f(x, \underline{v}(x)) \ge 0 \quad on \Omega \quad \overline{u} \le 0$$

Choose a number $c > 0$ such that

$$c + \tfrac{\partial f(x,u)}{\partial u} > 0 \quad \forall (x,u) \in \overline{\Omega} \times [\underline{v}, \overline{u}]$$

and such that the operator $(\Delta - c)$ with Dirichlet boundary condition has its spectrum strictly contained in the open left-half complex plane. Then the mapping

$$T : \phi \to w, \quad w = T\phi, \quad \phi \in C^2(\overline{\Omega}), \quad \phi(x) \in [\underline{v}, \overline{u}], \quad \forall x \in \overline{\Omega} \quad (3.1)$$

where $w(x)$ is the unique solution of the BVP

$$\begin{cases} \Delta w(x) - cw(x) = -[c\phi(x) + f(x, \phi(x))] & on \;\; \Omega \\ w(x) = 0 & on \;\; \partial\Omega, \end{cases} \tag{4}$$

is monotone, i.e. for any ϕ_1, ϕ_2 satisfying (3.1) and $\phi_1 \le \phi_2$, we have $T\phi_1, T\phi_2$ satisfies (3.1), and $T\phi_1 \le T\phi_2$ on Ω.

Consequently, by letting $f_c(x,u) = cu + f(x,u)$, the iterations

$$\begin{cases} u_0(x) = \overline{u}(x) & \\ (\Delta - c)u_{n+1}(x) = -f_c(x, u_n(x)) & on \ \Omega, \\ & n = 0,1,2,\ldots \\ u_{n+1}(x) = 0 & on \ \partial\Omega, \end{cases} \tag{5}$$

and

$$\begin{cases} v_0(x) = \underline{v}(x) & \\ (\Delta - c)v_{n+1}(x) = -f_c(x, v_n(x)) & on \ \Omega, \\ & n = 0,1,2,\ldots \\ v_{n+1}(x) = 0 & on \ \partial\Omega, \end{cases} \tag{6}$$

yield iteration u_n and v_n satisfying

$$\underline{v} = v_0 \le v_1 \le \cdots \le v_n \le \cdots \le u_n \le \cdots \le u_1 \le u_0 = \overline{u},$$

so that the limits

$$u_\infty(x) = \lim_{n\to\infty} u_n(x), \qquad v_\infty(x) = \lim_{n\to\infty} v_n(x)$$

exists in $C^2(\overline{\Omega})$. We have

(i) $v_\infty(x) \le u_\infty(x)$ on $\overline{\Omega}$

(ii) u_∞ and v_∞ are, respectively, stable from above and below;

(iii) If $u_\infty \not\equiv v_\infty$ and both u_∞ and v_∞ are asymptotically stable, then there exists an unstable solution $\phi \in C^2(\overline{\Omega})$ such that $v_\infty \le \phi \le u_\infty$

We use following algorithm

sub- and super-solution algorithm

. Find a subsolution v_0 and a supersolution u_0. Choose a number $c > 0$;

. Solve the boundary value problem

$$\begin{cases} -\Delta w_{n+1}(x) - c w_{n+1}(x) = -f_c(x, w_n(x)) & on\Omega \\ u(x) = 0 & on\partial\Omega. \end{cases} \tag{7}$$

for $w_n = v_n$ and $w_n = u_n$, respectively;

. If $\|w_{n+1} - w_n\| < \varepsilon$, output and stop. Else go to step 2.

We consider the problem $-\Delta u = \lambda g(x) f(u)$ with $g(x,y) = \frac{1}{2} - xy$, $\Omega = [0,1]\times[0,1]$ and $f(u) = \frac{1}{6}u(1-u)(u+\frac{5}{2})$. Also we will use the notation $\mathbf{u}$ to represent an array of real numbers agreeing with u on a grid $\Omega \subset \overline{\Omega}$. We will take the grid to be regular.

The obtained results shows there is an array of solution that before λ_1^+ is identically zero and after it has the norm less than the horizontal asymptote 1 when we define

$$\| u \| = \| u \|_\infty = \sup_{x\in[0,1]} u(x)$$

(see the following tables).

For brevity we express just some of those numerical results.

Approximation of u for $\lambda = 1$

x/y	0.1	0.3	0.5	0.7	0.9
0.1	0.0101×10^{-4}	0.1293×10^{-4}	-0.2697×10^{-4}	-0.2176×10^{-4}	-0.072×10^{-4}
0.3	0.1293×10^{-4}	-0.5956×10^{-4}	-0.6934×10^{-4}	-0.2867×10^{-4}	0.0204×10^{-4}
0.5	-0.2697×10^{-4}	-0.6934×10^{-4}	-0.3443×10^{-4}	-0.0210×10^{-4}	0.0168×10^{-4}
0.7	-0.2176×10^{-4}	-0.2867×10^{-4}	-0.0210×10^{-4}	-0.0255×10^{-4}	0.0200×10^{-4}
0.9	-0.072×10^{-4}	0.0204×10^{-4}	0.0168×10^{-4}	0.0200×10^{-4}	0.0090×10^{-4}

Approximation of u for $\lambda = 10$

x/y	0.1	0.3	0.5	0.7	0.9
0.1	0.0121×10^{-4}	-0.0969×10^{-4}	-0.1893×10^{-4}	-0.1444×10^{-4}	-0.0181×10^{-4}
0.3	-0.0969×10^{-4}	-0.4013×10^{-4}	-0.4038×10^{-4}	-0.0876×10^{-4}	0.0115×10^{-4}
0.5	-0.1893×10^{-4}	-0.4038×10^{-4}	0.1241×10^{-4}	0.0646×10^{-4}	0.0050×10^{-4}
0.7	-0.1444×10^{-4}	-0.0876×10^{-4}	-0.0649×10^{-4}	-0.0166×10^{-4}	0.0032×10^{-4}
0.9	-0.0181×10^{-4}	-0.0115×10^{-4}	-0.0050×10^{-4}	0.0032×10^{-4}	0.0033×10^{-4}

Approximation of super-solution $\overline{u}$ for $\lambda = 170$

x/y	0.1	0.3	0.5	0.7	0.9
0.1	0.0101	0.0239	0.0256	0.0179	0.0062
0.3	0.0239	0.0552	0.0570	0.0385	0.0130
0.5	0.0256	0.0570	0.0560	0.0359	0.0116
0.7	0.0179	0.0385	0.0359	0.0218	0.0067
0.9	0.0062	0.0130	0.0116	0.0067	0.0020

$\|u\|_\infty = 0.0570$

Approximation of super-solution $\underline{v}$ for $\lambda = 170$

x/y	0.1	0.3	0.5	0.7	0.9
0.1	0.0097	0.0231	0.0247	0.0173	0.0060
0.3	0.0231	0.0552	0.0551	0.0372	0.0125
0.5	0.0247	0.0551	0.0542	0.0347	0.0112
0.7	0.0173	0.0372	0.0347	0.0211	0.0065
0.9	0.0060	0.0125	0.0112	0.0065	0.0019

$\|v\|_\infty = 0.0551$

Approximation of super-solution $\overline{u}$ for $\lambda = 1000$

x/y	0.1	0.3	0.5	0.7	0.9
0.1	0.4778	0.6997	0.7048	0.6641	0.3951
0.3	0.6997	0.9633	0.9692	0.9056	0.5056
0.5	0.7048	0.9692	0.9542	0.7906	0.3246
0.7	0.6641	0.9056	0.7906	0.4710	0.1284
0.9	0.3951	0.5056	0.3246	0.1284	0.0234

$\|u\|_\infty = 0.9692$

Approximation of super-solution $\underline{v}$ for $\lambda = 1000$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.4778	0.6997	0.7048	0.6640	0.3948
0.3	0.6997	0.9632	0.9690	0.9051	0.5050
0.5	0.7048	0.9690	0.9517	0.7888	0.3233
0.7	0.6640	0.9051	0.7888	0.4681	0.1238
0.9	0.3948	0.5050	0.3233	0.1238	0.0232

$\| v \|_{\infty} = 0.9690$

Approximation of super-solution $\bar{u}$ for $\lambda = 80000$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.9913	0.9955	0.9953	0.9950	0.9896
0.3	0.9955	1	1	1	0.9970
0.5	0.9953	1	1	0.9998	0.9237
0.7	0.9950	1	0.9998	0.7388	0.0037
0.9	0.9696	0.9970	0.9237	0.0037	0.0000

$\| u \|_{\infty} = 1$

Approximation of super-solution $\underline{v}$ for $\lambda = 80000$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.9913	0.9955	0.9953	0.9950	0.9896
0.3	0.9955	1	1	1	0.9970
0.5	0.9953	1	1	0.9998	0.9237
0.7	0.9950	1	0.9998	0.7298	0.0037
0.9	0.9896	0.9970	0.9237	0.0037	0.0000

$\| v \|_{\infty} = 1$

We guess that $\| u \| \to 1$ as $\lambda \to \infty$ and we guess that is less than 200.

3. References

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