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# Chaos Synchronization of Willis Aneurysm Systems *

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**Abstract.** In this paper an adaptive sliding mode controller is presented for a class of master-slave chaotic synchronization systems with uncertainties. The concept of extended systems is used such that continuous control input is obtained using a sliding mode design scheme. By comparing with the results in the existed literatures, the results show that the master-slave chaotic systems with uncertainties can be synchronized accurately by this controller. Illustrative examples of chaos synchronization for uncertain Willis system are presented to demonstrate the superiority of the obtained results.

**Keywords:** Willis Systems, Chaos, Sliding Control, Synchronization.

## 1. Introduction

An aneurysm is a localized dilatation of blood vessels caused by congenital, traumatic, arteriosclerotic or other factors. Congenital and traumatic aneurysms are most commonly found in cerebral blood vessels, and are a major cause of stroke-related morbidity and mortality. The pathogenesis of aneurysm formation and rupture is not clearly understood and it depends on many factors. Many papers consider the mathematical formulation of the blood flow some of them in relation to aneurysm. Different models of studying aneurysms have been considered in the literature. An important factor is the nonlinear nature of wall elasticity and some mathematical questions are considered [1]. Recently a modified nonlinear equation was introduced [2] to study the blood flow inside an aneurysm of the circle of Willis. A different biomechanical model of the flow in the circle of Willis is presented [3]. A two-dimensional nonlinear mathematical model is described [4] to study the aneurysm .now it is generally accepted that turbulence, chaos and fractals frequently appear in medicine [5]. Thus we note that turbulent flow is observed within an aneurysm, as evidenced by reduced bruits [6,7], where the inter action between the aneurysm vibration and the blood flow is recognized as having the characteristic features of a nonlinear feedback system, and existence of chaotic solutions [8]. Chaos synchronization has received increasing attention. Many methods have been presented for the synchronization of chaotic system such as periodic parametric perturbation method [9,10], drive-response synchronization method [11], adaptive control method [12,13], variable structure (or sliding mode) control method [14,15], backstepping control method [16], and $H_\infty$ control method [17]. Basically, the chaos synchronization problem means making two systems oscillate in a synchronized manner. Given a chaotic system, which is considered as the master system, and another identical system, which is considered as the slave system, the dynamical behaviors of these two systems may be identical after a transient when the slave system is driven by a control input.

In this paper, the organization of this paper is as follows. In Section 2, the master-slave chaos synchronization system is described; the proposed controller design methodology is presented in Section 3; Section 4 presents simulation results.

## 2. Nonlinear model of blood flow in aneurysm

Using the electric analogue of [18], and denoting by $i_2$, the velocity of blood flow inside aneurysm we get the following equation governing this velocity

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$$E' = R_1 R_2 C_1 \frac{d^2 i_2}{dt^2} + (\frac{R_1 R_2}{R_3} + R_1 + R_2)\frac{di_2}{dt} + R_1 C_1 \frac{d^2 e}{dt^2} + (\frac{R_1}{R_3} + 1)\frac{de}{dt}, \quad (1)$$

where $E'$ is central blood pressure; $I$ is velocity of blood flow in the parent blood vessel; $i$ is velocity of blood flow $i_1$, in blood vessel at site of aneurysm; is velocity of blood $i_2$, flow inside aneurysm; $C_1$ is elasticity of segment of vessel wall; $C_2$ is elasticity of aneurysm wall; $e$ is pressure in aneurysm; $R$ is resistance to flow; and $V$ is pressure in parent blood vessels. The expression for $e$ has been obtained experimentally in latex and rubber aneurysm models, and it is of the form $e = \int(\alpha i_2 - \beta i_2^2 + \gamma i_2^3)dt$ thus $e = \int \varphi(i_2(t))dt$ , $E' = F\cos(\omega t)$ represents the rate of change of the central blood pressure, $F$ is equivalent to the pulse pressure, and v is the inverse of the cardiac frequency. Note that any change in $F$ produces a change in both pressure and blood pressure. To simplify the notation, set, Hence Eq.(1) is now

$$\ddot{x} + p\dot{x} + ax - bx^2 + cx^3 + rx^2\dot{x} - qx\dot{x} = F\cos(wt) \quad (2)$$

with

$$p = \frac{\frac{R_1 R_2}{R_3} + R_1 + R_2 + \alpha R_1 R_3}{R_1 R_2 C_1}; a = \frac{\alpha(\frac{R_1}{R_3} + 1)}{R_1 R_2 C_1}; b = \frac{\beta(\frac{R_1}{R_3} + 1)}{R_1 R_2 C_1};$$

$$c = \frac{\gamma(\frac{R_1}{R_3} + 1)}{R_1 R_2 C_1}; \ \gamma = \frac{-2\beta R_1 C_1}{R_1 R_2 C_1}; \ q = \frac{-2\gamma R_1 C_1}{R_1 R_2 C_1}.$$

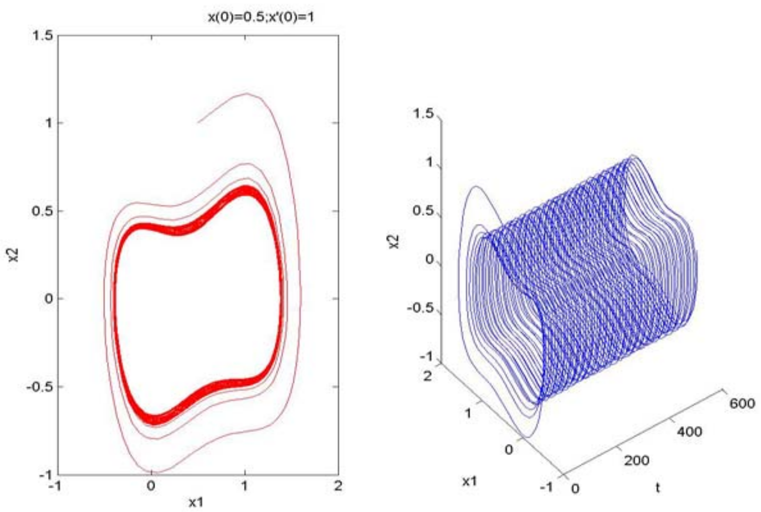


Fig.1. A particular solution of aneurysm equation with initial x(0)=0.5 ; x'(0)=1;r=1.5;q=2.5

For different initial conditions, both models depend on the velocity of the blood flow inside the aneurysm see Fig. 1, Fig. 2 and Lyapunov exponents see Fig. 3. Parameters are $a = 1, b = 3, c = 2, F = 0.01, p = 0.1, \omega = 1$.

Now consider a more general model: aneurysm model[19] that is, $q = r = 0$ in(2):

$$\ddot{x} + p\dot{x} + ax - bx^2 + cx^3 = F\cos(wt), (p,\ a,\ b,\ c > 0), \quad (3)$$

with $a, b, c, F$ positive constants, is a biomathematical model for the blood flow inside an aneurysm of the circle of Willis. For some medical questions related to this model, here $u$ represents the velocity of the blood flow inside the aneurysm. It is a second-order nonlinear ordinary differential equation with periodic forcing term. The mathematical analysis of this biomathematical model allows us to obtain some basic information on the evolution of the aneurysm. Our model is in accordance with some clinical observations. For instance, either an increment or a sudden change in the blood pressure leads to chaotic mathematical solutions and hence to turbulent flow inside the aneurysm, with a risk of rupture of the aneurysm. The same conclusion applies to an abrupt change of the cardiac frequency. For different initial conditions, both models depend on the velocity of the blood flow inside the aneurysm see Fig.4. Fig.5. Fig.6.and Lyapunov exponents see Fig.7. Parameters are $a = 1, b = 3, c = 2, F = 0.01, p = 0.1, \omega = 1$.

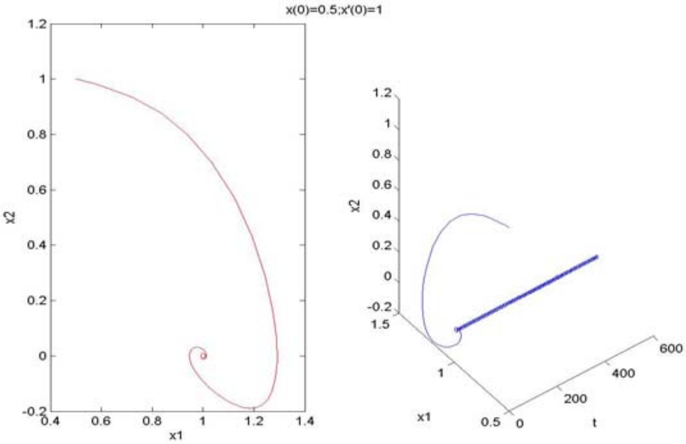


Fig.2. A particular solution of aneurysm equation with initial x(0)=0.5 ; x'(0)=1;r=1.5;q=0.5

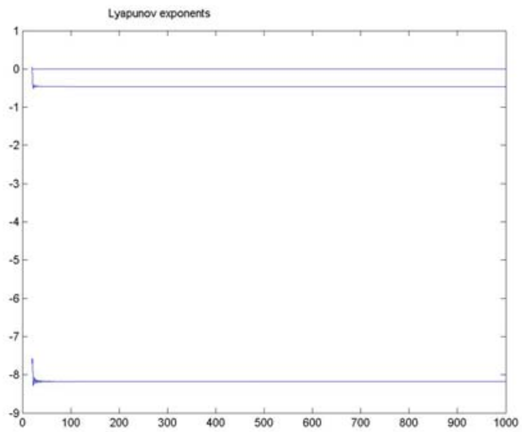


Fig.3. Lyapunov exponents of Willis system (LE1=0; LE2=-0.46225;LE3=-8.1788;LD=0)

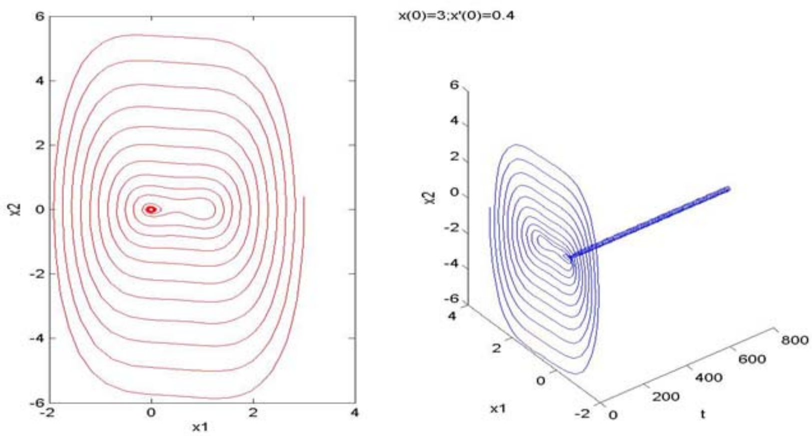


Fig.4. A particular solution of aneurysm equation with initial x(0)=3 and x'(0)=0.4

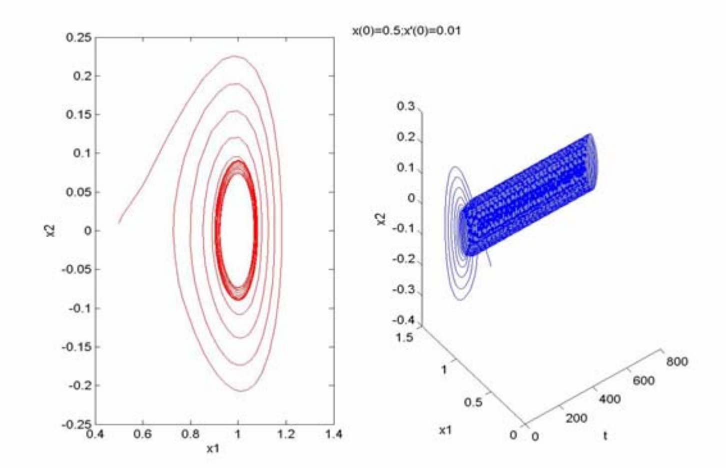


Fig.5. A particular solution of aneurysm equation with initial x(0)=0.5 and x'(0)=0.01

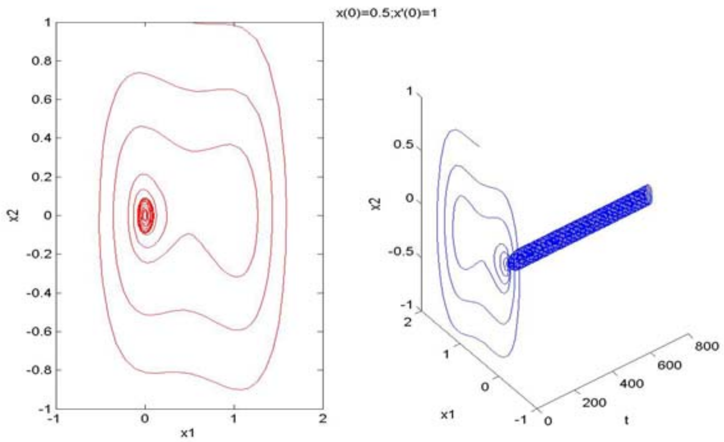


Fig.6. A particular solution of aneurysm equation with initial x(0)=0.5 and x'(0)=1

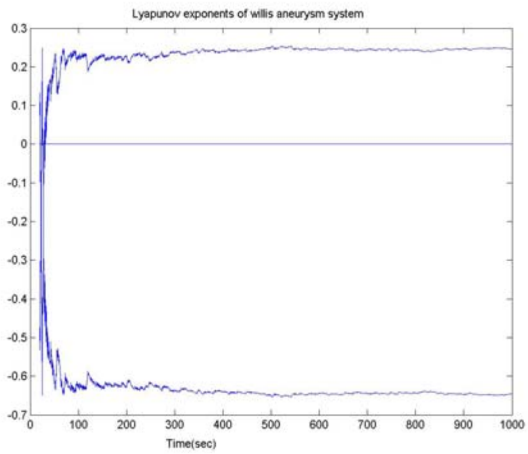


Fig.7. Lyapunov exponents of Willis system (LE1=0.24662;LE2=0;LE3=-0.64662;LD=2.3814)

The formation and rupture of aneurysms is a significant medical problem, but is not clearly understood. Most intracranial aneurysm is located in the circle of Willis. We consider a nonlinear mathematical model that simulates the blood flow inside the aneurysm, one of the relevant factors in the evolution of an aneurysm. Different nonlinear analysis like chaos would help to understand some medical aspects of aneurysms of the circle of Willis. In order to improve the performance of the dynamic of the system or avoid the chaotic phenomena, we need to control a chaotic system to a periodic motion which is beneficial foe working with particular condition. Very recently much interest has been focused on this type of problem, controlling chaos, Chaos synchronization is consider for medical problem in this paper.

## 3. Sliding model control

Consider the following chaotic systems:

$$\dot{x} = f(x) + Bu \tag{4}$$

$x \in R^n$ is the state vector, $f : R^n \to R^n$ is the nonlinear function, $B \in R^n$ is the input matrix ,and $u \in R$ is the control signal. We assume that the system(3) behaves chaotically without control. The unstable fixed point of the chaotic system $x_f$ satisfies $f(x_f) = 0$.

We introduce a state vector $y = x - x_f$, and then the chaotic system (4) can be written as

$$\dot{y} = g(y) + Bu \tag{5}$$

We divide the function $g$ into the two parts as follows,

$$\dot{y} = Ay + h(y) + Bu \tag{6}$$

where $Ay$ is the linear part and $h(y)$ is the nonlinear part of $g(y)$.

The control problem considered in this paper is that for different initial conditions of systems(4) and (5),the two coupled system,i.e.the master systems(4) and the slave systems(5),to be synchronized by designing an appropriate control $u(t)$ which is attached to the slave systems(5) such that $\lim_{t\to\infty}\|x(t) - y(t)\| \to 0$,where $\|\cdot\|$ is the Euclidean norm of a vector.

The controller decides the signal depending on the switching function $\delta = sy$ $s \in R^{1\times n}$,the condition $\delta = 0$ indicates the sliding surface $\Gamma = \{y : sy = 0\} \subset R^n$,the sign of $\delta$ decides the control signal $u$.if the controlled orbit is in the sliding model, the following condition is satisfied $\delta = \dot{\delta} = 0$.we use Lyapunov function designs a sliding controller that guarantees to keep the orbit being the sliding model.

## 4. Adaptive sliding model controller design

Let the error state be $e_i = y_i - x_i, i = 1,2,\cdots,n$,and $g(e,t) = f(e + x,t) - f(x,t)$,the error dynamic equations is

$$\begin{aligned} \dot{e}_i &= e_{i+1};\ 1 \le i \le n-1, \\ \dot{e}_n &= \dot{y}_n - \dot{x}_n = f(y,t) - f(x,t) + \Delta f(y) + d(t) + u(t) \\ &= g(e,t) + \Delta f(e + x) + d(t) + u(t), \end{aligned} \tag{7}$$

Using the concept of extended systems, the standardized state space equations of the error states can be obtained as

$$\begin{aligned} \dot{e}_i &= e_{i+1};\ 1 \le i \le n-1, \\ \dot{e}_n &= g(e,t) + \Delta f(e + x) + d(t) + u(t) = e_{n+1}, \\ \dot{e}_{n+1} &= \frac{d}{dt}(g(e,t) + \Delta f(e + x) + d(t)) + \dot{u}(t), \end{aligned} \tag{8}$$

Based on the control law proposed by Chen and Lin [19], the sliding surface can be defined as

$$s = e_{n+1} - e_{n+1}(0) + \int_0^t \sum_{j=1}^{n+1} c_j e_j \, dt = 0, \tag{9}$$

where $e_{n+1}(0)$ denotes the initial state of $e_{n+1}$,Eq.(8) can also be formulated as

$$\dot{e}_{n+1} = -\sum_{j=1}^{n+1} c_j e_j \ ,$$

with initial condition $e_{n+1}(0) = e_{0(n+1)}$ and the sliding mode dynamics can be described by the following system of equations:

$$\dot{e}_1 = e_2,$$
$$\dot{e}_2 = e_3,$$
$$\cdots\cdots\cdots\cdots\cdots..$$
$$\dot{e}_{n+1} = -(c_1 e_1 + c_2 e_2 + \cdots + c_{n+1} e_{n+1}) \tag{10}$$

or in a matrix equation form as

$$\dot{e}_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ -c_1 & -c_2 & \cdots & \cdots & \cdots & -c_{n+1} \end{bmatrix} e_i = A_i e_i, \ 1 \le i \le n+1, \tag{11}$$

where

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ -c_1 & -c_2 & \cdots & \cdots & \cdots & -c_{n+1} \end{bmatrix}$$

with the initial states being $e_i(0) = [e_{0(1)}, e_{0(2)}, \cdots, e_{0(n+1)}]^T$ The design parameters $c_j$ can be determined by choosing the eigenvalues of $A_i$ such that the corresponding characteristic polynomial.

$$P(e) = \dot{e}_{n+1} + \sum_{j=1}^{n+1} c_j e_j \tag{12}$$

Let the control law be represented as:

$$u = u_{eq} + u_{sw}$$

where $u_{eq}$ is the equivalent control and $u_{sw}$ is the switching control. Suppose the approaching speed can be described by an adaptive law, then the reaching law can be chosen such that

$$\dot{s} = -(\hat{\beta} + k)sign(s), \tag{13}$$

where $sign(\cdot)$ denote the sign function , $k$ is positive constant value and $\hat{\beta}$ is estimated parameter which satisfies the following adaptive

$$\dot{\hat{\beta}} = |s|, \quad \hat{\beta}(0) = \hat{\beta}_0 \tag{14}$$

where $\hat{\beta}_0$ is the bounded positive initial condition of $\hat{\beta}$.

From Eqs.(9) and(13),it can be found that

$$\begin{aligned} \dot{s} = \dot{e}_{n+1} + \sum_{j=1}^{n+1} c_j e_j &= \frac{d}{dt}(g(e,t) + \Delta f(e+x) + d(t)) + u(t) \\ &= -(\hat{\beta} + k)sign(s) - \sum_{j=1}^{n+1} c_j e_j \end{aligned} \tag{15}$$

the differential equation of control signal

$$\dot{u}(t) = -\frac{d}{dt}(g(e,t) + \Delta f(e+x) + d(t)) - (\hat{\beta} + k)sign(s) - \sum_{j=1}^{n+1} c_j e_j \quad (16)$$

the system uncertainty $\Delta f(e+x)$ and the external disturbance $d(t)$ are unknown and implemented control input is described by

$$\dot{u}(t) = -\frac{d}{dt}(g(e,t) - (\hat{\beta} + k)sign(s) - \sum_{j=1}^{n+1} c_j e_j \quad (17)$$

## 5. Adaptive synchronization of Willis aneurysm systems

Willis aneurysm systems(3), Our goal is to control the system output x tracking the reference signal $x_d$, therefore, the problem is to design a controller $u(t)$ so that item $x - x_d$ converges to zero.

To see the effectiveness of the proposed method and ideal, let us look at the master-slave Willis system. Consider two coupled Willis systems as follows: Let $x_1 = x_1$, $x_2 = \dot{x}_1 = \dot{x}$, the equation above is equal to

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -px_2 - ax_1 + bx_1^2 - cx_1^3 + F\cos(wt) \end{cases} \quad (18)$$

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -py_2 - ay_1 + by_1^2 - cy_1^3 + \Delta f(y) + d(t) + F\cos(wt) + u(t) \end{cases} \quad (19)$$

The control input $u(t)$ is attached to the second equation of the slave system (13) and the slave system is perturbed by an uncertainty term $\Delta f(y)$ and interfered with a disturbance $d(t)$, added to its second equation. Let the error states are $e_1 = y_1 - x_1$ $e_2 = y_2 - x_2$, Subtracting (18) from (19) yields the synchronization error dynamics as:

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = -pe_2 - ae_1 + b(y_1^2 - x_1^2) - c(y_1^3 - x_1^3) + \Delta f(e+x) + d(t) + u(t) \end{cases} \quad (20)$$

Then, the standardized state space equations can be described as

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = e_3 \\ \dot{e}_3 = -pe_3 - ae_2 + 2b(y_1\dot{y}_1 - x_1\dot{x}_1) - 3c(y_1^2\dot{y}_1 - x_1^2\dot{x}_1) + \frac{d}{dt}(\Delta f(e+x) + d(t)) + \dot{u}(t) \end{cases} \quad (21)$$

Let the sliding surface be defined as

$$s = e_3 - e_3(0) + \int_0^t (c_3e_3 + c_2e_2 + c_1e_1)dt \quad (22)$$

The eigenvalues corresponding to the sliding surface can be decided by and these eigenvalues dominate the converging rate of the error dynamics and they can arbitrarily be assigned. Choose the reaching law as in Eq. (3). From Eqs. (3), (21) and (22), the control input is determined as

$$u = \int_0^t \{pe_3 + ae_2 - 2b(y_1\dot{y}_1 - x_1\dot{x}_1) + 3c(y_1^2\dot{y}_1 - x_1^2\dot{x}_1) - (c_3e_3 + c_2e_2 + c_1e_1) - (\hat{\gamma} + k)sign(s)\}dt$$

$$\dot{u}(t) = pe_3 + ae_2 - 2b(y_1\dot{y}_1 - x_1\dot{x}_1) + 3c(y_1^2\dot{y}_1 - x_1^2\dot{x}_1) - (c_3e_3 + c_2e_2 + c_1e_1) - (\hat{\gamma} + k)sign(s)$$

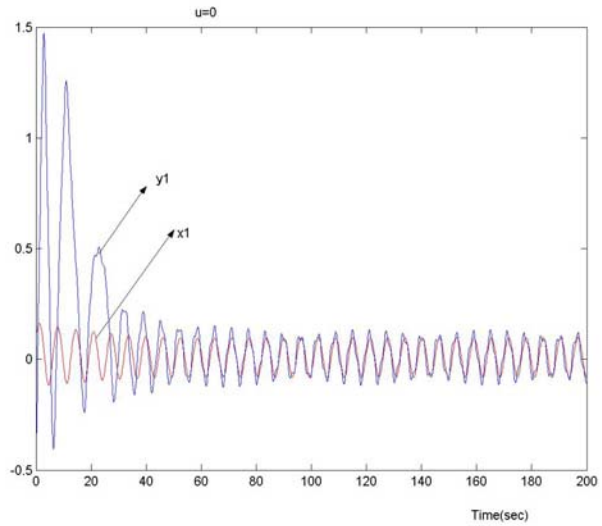


Fig. 8. Time responses of Willis chaos synchronization: master and slave system outputs are x1 and y1, respectively.

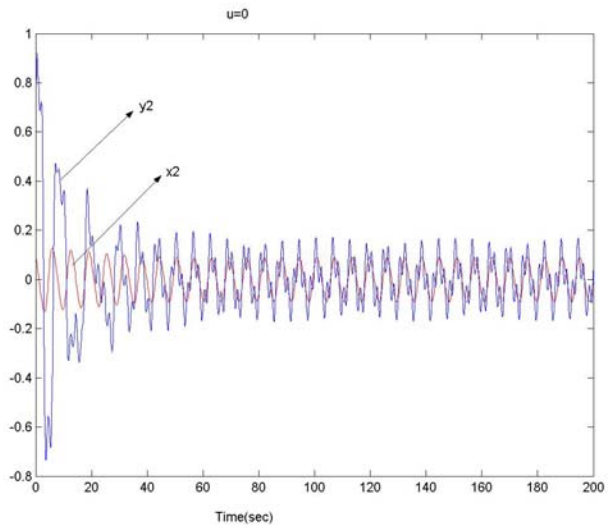


Fig. 9. Time responses of Willis chaos synchronization: master and slave system outputs are x2 and y2, respectively.

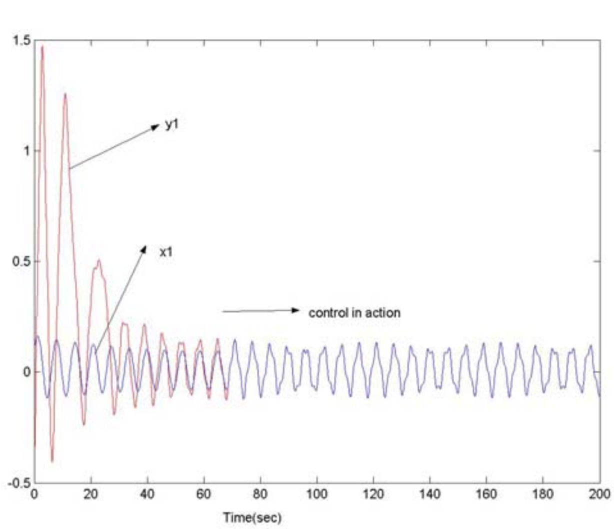


Fig. 10. Time responses of Willis chaos synchronization: master and slave system outputs are x1 and y1, respectively. The control u(t) is activated at 60 s.

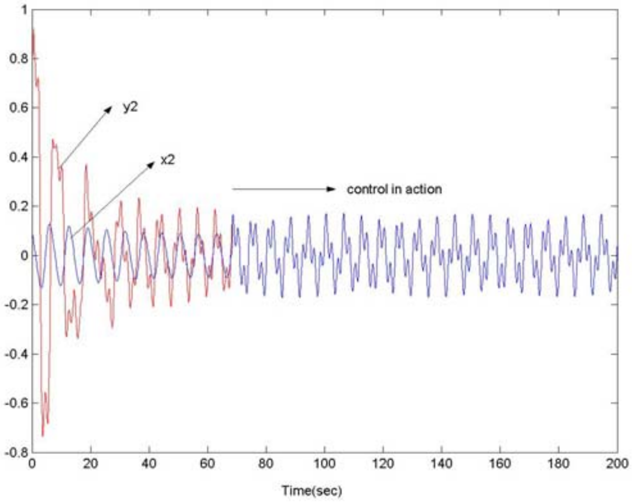


Fig.11. Time responses of Willis chaos synchronization: master and slave system outputs are x2 and y2, respectively. The control u(t) is activated at 60 s

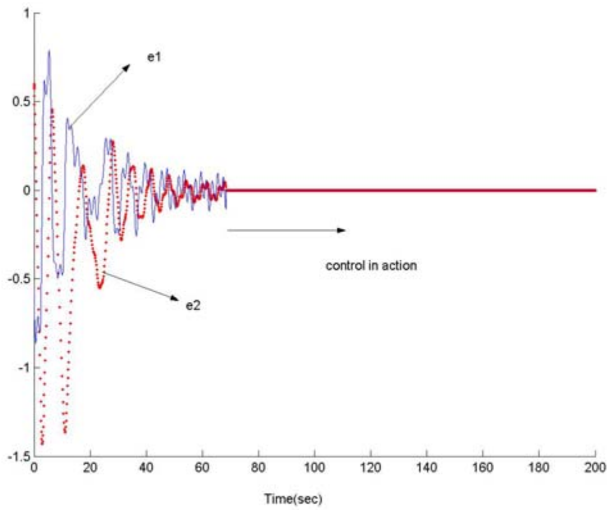


Fig. 12. The time response of error states. The control u(t) is activated at 20 s.

## 6. Conclusion

This paper presents a method to design an adaptive sliding mode controller for aneurysm model chaos synchronization with system uncertainties and disturbance. Based on the Lyapunov stability theory, an adaptive sliding mode controller is designed for the regulation of the error state vector to a desired point in the state space. To design the proposed control scheme, the requirement of the bound information of the uncertainties is not need. According to the simulations, the proposed method can be successfully applied to synchronization problems of aneurysm model chaos.

The derived controllers are robust so that the closed-loop system is stable in the presence of uncertainties and disturbance.

The chattering phenomenon of conventional switching type sliding controls does not occur in this study.

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