ISSN 1746-7659, England, UK
Journal of Information and Computer Science
Vol. 1, No. 4, 2006, pp. 195-206.

On the Properties of Credibility Critical Value Functions

Yan-Kui Liu* , Shuming Wang

College of Mathematics and Computer Science, Hebei University, Baoding 071002, Hebei, China

(*Received Nov. 20, 2005, Accepted March 12, 2006*)

Abstract. The objective of this paper is to deal with the analytical properties of credibility critical value functions. Based on the continuity of credibility functions, several sufficient and necessary conditions for some properties of the credibility critical value functions are established. The results obtained in this paper can be used to fuzzy optimization problems as we design algorithms to compute the critical values of objective functions.

Keywords: fuzzy variable, credibility function, critical value function; sufficient and necessary condition

1 Introduction

Since Zadeh's work [17], possibility theory was being perfected and extended by many researchers such as [1, 3, 12, 15, 16], it became a strong tool to deal with possibilistic uncertainty. In literature, many successful applications can be found in the field of fuzzy optimization such as [2, 8, 9, 10, 14]. Based on possibility measure, a self-dual set function, called credibility measure was introduced in [6], which can be regarded as the counterpart of probability measure in fuzzy decision systems. Based on credibility measure, an axiomatic approach, called credibility theory [7], was studied extensively. From a measure-theoretic viewpoint, credibility theory provides a theoretical foundation for fuzzy programming [8], just like the role of probability theory in stochastic programming [4].

Since credibility measure is a nonadditive set function, the credibility functions related to a fuzzy variable ξ, $\mathrm{Cr}\{\xi \geq x\}$ and $\mathrm{Cr}\{\xi \leq x\}$, are neither left continuous nor right continuous [7]. Recently, some analytical properties of the credibility functions have been discussed in [5], and the sufficient and necessary conditions for the continuity of $\mathrm{Cr}\{\xi \geq x\}$ and $\mathrm{Cr}\{\xi \leq x\}$, were established, respectively. Based on the continuity of the credibility functions, our goal in this paper is to discuss the properties of credibility critical value function of a fuzzy variable.

This paper is organized as follows. In Section 2, we recall some basic concepts in credibility theory and some results on the credibility functions which will be used in the rest of the paper. Section 3 deals with the properties of the credibility optimistic value functions. In this section, we first discuss the conditions under which the inequalities $\mathrm{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \geq \alpha$, $\mathrm{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \leq \alpha$ hold for any $\alpha \in (0, 1]$. Then we establish a sufficient and necessary condition for the equality $\mathrm{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha$. At last, a sufficient and necessary condition for the continuity of $\xi_{\sup}(\alpha)$ with respect to α is obtained. The corresponding results of the credibility pessimistic value functions are covered in Section 4. Finally, a brief summary is given in Section 5.

* Corresponding author. Tel: +86-312-5066629.
E-mail address: yliu@mail.hbu.edu.cn (Y.-K. Liu).

2 Preliminaries

Let ξ be a fuzzy variable with possibility distribution $\mu : \Re \rightarrow [0,1]$. It is well known that for any $x \in \Re$, the possibility of the event $\{\xi \geq x\}$ is defined by

$$\mathrm{Pos}\{\xi \geq x\} = \sup_{t \geq x} \mu(t). \tag{1}$$

Definition 2.1 ([6]) *Let ξ be a fuzzy variable. The credibility of the event $\{\xi \geq x\}$ is defined as*

$$\mathrm{Cr}\{\xi \geq x\} = \frac{1}{2}(1 + \mathrm{Pos}\{\xi \geq x\} - \mathrm{Pos}\{\xi < x\}). \tag{2}$$

Definition 2.2 ([7]) *Let ξ be a fuzzy variable, $\alpha \in (0,1]$. The α-optimistic value function of ξ is defined as*

$$\xi_{\sup}(\alpha) = \sup\{x \mid \mathrm{Cr}\{\xi \geq x\} \geq \alpha\}, \tag{3}$$

while the α-pessimistic value function of ξ is defined as

$$\xi_{\inf}(\alpha) = \inf\{x \mid \mathrm{Cr}\{\xi \leq x\} \geq \alpha\}. \tag{4}$$

Let $f(x)$ be a real valued function defined on $\Re$. Then we say $f(x)$ is upper semicontinuous at x_0 if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $f(x) < f(x_0) + \varepsilon$ for all x with $|x - x_0| < \delta$. We say $f(x)$ is lower semicontinuous at x_0 if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $f(x) > f(x_0) - \varepsilon$ for all x with $|x - x_0| < \delta$.

In addition, we have

(a) $\limsup_{x \to x_0} f(x) = \inf_{\delta > 0} \sup_{0 < |x - x_0| < \delta} f(x)$,

(b) $\limsup_{x \to x_0+} f(x) = \inf_{\delta > 0} \sup_{0 < x - x_0 < \delta} f(x)$,

(c) $\limsup_{x \to x_0-} f(x) = \inf_{\delta > 0} \sup_{-\delta < x - x_0 < 0} f(x)$.

More detailed properties about the upper and lower semicontinuity can be found in [11, 13].

Definition 2.3 *Let ξ be a fuzzy variable with possibility distribution $\mu(x)$. Then we say ξ is right continuous (resp., left continuous, upper semicontinuous or lower semicontinuous) if $\mu(x)$ is right continuous (resp., left continuous, upper semicontinuous or lower semicontinuous).*

Theorem 2.1 ([5]) *Let ξ be a fuzzy variable with possibility distribution $\mu(x)$. Then $\mathrm{Cr}\{\xi \geq x\}$ is left continuous if and only if for any $x \in \Re$, $\limsup_{t \to x-} \mu(t) \leq \mathrm{Pos}\{\xi \geq x\}$.*

Theorem 2.2 ([5]) *Let ξ be a fuzzy variable with possibility distribution $\mu(x)$. Then $\mathrm{Cr}\{\xi \leq x\}$ is right continuous if and only if for any $x \in \Re$, $\limsup_{t \to x+} \mu(t) \leq \mathrm{Pos}\{\xi \leq x\}$.*

Theorem 2.3 ([5]) *If fuzzy variable ξ is both right continuous and lower semicontinuous, then the credibility function $\mathrm{Cr}\{\xi \geq x\}$ is right continuous.*

Theorem 2.4 ([5]) *If fuzzy variable ξ is both left continuous and lower semicontinuous, then $\mathrm{Cr}\{\xi \leq x\}$ is left continuous.*

Theorem 2.5 ([5]) *Let ξ be a lower semicontinuous fuzzy variable with possibility distribution $\mu(x)$. If*

$$\limsup_{x \to x_0+} \mu(x) \leq \mathrm{Pos}\{\xi \leq x_0\}, \textit{ and } \limsup_{x \to x_0-} \mu(x) \leq \mathrm{Pos}\{\xi \geq x_0\},$$

then both $\mathrm{Cr}\{\xi \geq x\}$ and $\mathrm{Cr}\{\xi \leq x\}$ are continuous at x_0.

3 The Properties of Optimistic Value Functions

The intend of this section is to discuss the properties of the credibility optimistic value functions based on the continuity of the credibility function $\text{Cr}\{\xi \geq x\}$.

Proposition 3.1 *Let ξ be a fuzzy variable. Then for any $\alpha \in (0,1]$, $\xi_{\sup}(\alpha) < \infty$ if and only if $\lim_{x\to\infty} \text{Cr}\{\xi \geq x\} = 0$.*

Proof. *Sufficiency:* We use a proof by contradiction. Suppose there exists an $\alpha_0 \in (0,1]$ such that for any large number $M > 0$, we have

$$\xi_{\sup}(\alpha_0) = \sup\{x \mid \text{Cr}\{\xi \geq x\} \geq \alpha_0\} > M,$$

then we can always find an x_0 with $M < x_0 < \xi_{\sup}(\alpha_0)$ such that $\text{Cr}\{\xi \geq x_0\} \geq \alpha_0$, which is a contradiction with $\lim_{x\to\infty} \text{Cr}\{\xi \geq x\} = 0$. The sufficiency is proved.

Necessity: We also use a proof by contradiction. Suppose $\lim_{x\to\infty} \text{Cr}\{\xi \geq x\} \neq 0$, i.e., there is a $\varepsilon_0 > 0$, for any N, we can find an $x_0 > N$ such that $\text{Cr}\{\xi \geq x_0\} > \varepsilon_0$. If we take $\alpha_0 = \min\{\varepsilon_0, \frac{1}{2}\} \in (0,1]$, then for any N, there is an $x_0 > N$ such that $\text{Cr}\{\xi \geq x_0\} > \alpha_0$. It follows from

$$\xi_{\sup}(\alpha_0) = \sup\{x \mid \text{Cr}\{\xi \geq x\} \geq \alpha_0\}$$

that $\xi_{\sup}(\alpha_0) \geq x_0 > N$, which is a contradiction with that for any $\alpha \in (0,1]$, $\xi_{\sup}(\alpha)$ is finite. The necessity is proved.

In this section, we always assume that fuzzy variable ξ satisfies $\lim_{x\to\infty} \text{Cr}\{\xi \geq x\} = 0$. Thus, by Proposition 3.1, for any $\alpha \in (0,1]$, $\xi_{\sup}(\alpha) < \infty$.

3.1 On Inequalities $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \geq \alpha$ and $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \leq \alpha$

Theorem 3.1 *Let ξ be a fuzzy variable. Then we have*

(i) $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \geq \alpha$ provided that $\text{Cr}\{\xi \geq x\}$ is left continuous at $\xi_{\sup}(\alpha)$;

(ii) $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \leq \alpha$ provided that $\text{Cr}\{\xi \geq x\}$ is right continuous at $\xi_{\sup}(\alpha)$.

Proof. We only prove assertion (i), assertion (ii) can be proved in a similar way. On the one hand, denote

$$z = \xi_{\sup}(\alpha) = \sup\{x \mid \text{Cr}\{\xi \geq x\} \geq \alpha\}.$$

Then, for any $x < z$, we have $\text{Cr}\{\xi \geq x\} \geq \alpha$.

On the other hand, since $\text{Cr}\{\xi \geq x\}$ is left continuous at z, i.e.,

$$\lim_{x\to z-} \text{Cr}\{\xi \geq x\} = \text{Cr}\{\xi \geq z\}.$$

Thus, for any $\varepsilon > 0$, there exist $\delta > 0$ and x with $-\delta < x - z < 0$ such that

$$\text{Cr}\{\xi \geq z\} > \text{Cr}\{\xi \geq x\} - \varepsilon \geq \alpha - \varepsilon.$$

It follows from the arbitrary of ε that $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = \text{Cr}\{\xi \geq z\} \geq \alpha$. The proof of the proposition is complete.

Theorem 3.2 *Let ξ be a fuzzy variable.*

(i) $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \geq \alpha$ for every $\alpha \in (0,1]$ if and only if $\text{Cr}\{\xi \geq x\}$ is left continuous;

(ii) $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \leq \alpha$ *for every* $\alpha \in (0,1]$ *if and only if* $\text{Cr}\{\xi \geq x\}$ *is right continuous.*

Proof. We only prove assertion (i), and assertion (ii) can be similarly proved.

Sufficiency: Since $\text{Cr}\{\xi \geq x\}$ is left continuous, for any $\alpha \in (0,1]$, $\text{Cr}\{\xi \geq x\}$ is left continuous at $\xi_{\sup}(\alpha)$. It follows from Theorem 3.1 that $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \geq \alpha$. This proves the sufficiency.

Necessity: We use a proof by contradiction. Suppose there is an x_0 such that $\text{Cr}\{\xi \geq x\}$ is not left continuous at x_0, then

$$\lim_{x \to x_0 -} \text{Cr}\{\xi \geq x\} > \text{Cr}\{\xi \geq x_0\}.$$

Take $\alpha_0 = \lim_{x \to x_0 -} \text{Cr}\{\xi \geq x\}$. Then we have $\text{Cr}\{\xi \geq x_0\} < \alpha_0$. If we deduce

$$x_0 = \xi_{\sup}(\alpha_0) = \sup\{x \mid \text{Cr}\{\xi \geq x\} \geq \alpha_0\}, \tag{5}$$

then, $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha_0)\} < \alpha_0$, which is a contradiction with that for any $\alpha \in (0,1]$, $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \geq \alpha$. We now prove Eq. (5) in the following, on the one hand, noting that $\text{Cr}\{\xi \geq x_0\} < \alpha_0$, one has $\text{Cr}\{\xi \geq x\} \leq \text{Cr}\{\xi \geq x_0\} < \alpha_0$ for any $x \geq x_0$. That is if $\text{Cr}\{\xi \geq x\} \geq \alpha_0$, then $x < x_0$.

On the other hand, for any $\varepsilon > 0$, there is an x^* such that $x_0 - \varepsilon < x^* < x_0$ and $\text{Cr}\{\xi \geq x^*\} \geq \lim_{x \to x_0 -} \text{Cr}\{\xi \geq x\} = \alpha_0$. Thus, by the definition of $\xi_{\sup}(\alpha_0)$, we have

$$x_0 = \xi_{\sup}(\alpha_0) = \sup\{x \mid \text{Cr}\{\xi \geq x\} \geq \alpha_0\}.$$

The necessity is proved.

Theorem 3.3 *Let* ξ *be a fuzzy variable. Then* $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \geq \alpha$ *for every* $\alpha \in (0,1]$ *if and only if for any* $x \in \Re$, $\limsup_{t \to x-} \mu(t) \leq \text{Pos}\{\xi \geq x\}$.

Proof. *Sufficiency:* Suppose for any $x \in \Re$, $\limsup_{t \to x-} \mu(t) \leq \text{Pos}\{\xi \geq x\}$. By Theorem 2.1, $\text{Cr}\{\xi \geq x\}$ is left continuous. It follows from (1) in Theorem 3.2 that for every $\alpha \in (0,1]$, we have $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \geq \alpha$. The sufficiency is proved.

Necessity: Suppose $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \geq \alpha$ for every $\alpha \in (0,1]$, It follows from (1) in Theorem 3.2 that $\text{Cr}\{\xi \geq x\}$ is left continuous. Therefore, by Theorem 2.1, for any $x \in \Re$, $\limsup_{t \to x-} \mu(t) \leq \text{Pos}\{\xi \geq x\}$. The necessity is proved.

Theorem 3.4 *If fuzzy variable* ξ *is both right continuous and lower semicontinuous, then* $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \leq \alpha$ *for every* $\alpha \in (0,1]$.

Proof. Since ξ is a both right continuous and lower semicontinuous fuzzy variable, by Theorem 2.3, $\text{Cr}\{\xi \geq x\}$ is right continuous. It follows from (2) in Theorem 3.2 that $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \leq \alpha$ for every $\alpha \in (0,1]$.

3.2 On Equality $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha$

Corollary 3.1 *If* ξ *is a fuzzy variable, and* $\text{Cr}\{\xi \geq x\}$ *is continuous at* $\xi_{\sup}(\alpha)$, *then* $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha$.

Proof. The corollary follows immediately from Theorem 3.2.

Theorem 3.5 *Let* ξ *be a lower semicontinuous fuzzy variable with possibility distribution* $\mu(x)$, *and* $a_0 = \xi_{\sup}(\alpha)$. *If*

$$\limsup_{x \to a_0 +} \mu(x) \leq \text{Pos}\{\xi \leq a_0\}, \text{ and } \limsup_{x \to a_0 -} \mu(x) \leq \text{Pos}\{\xi \geq a_0\},$$

then $\text{Cr}\{\xi \geq a_0\} = \alpha$.

Proof. By Theorem 2.5, we know that $\text{Cr}\{\xi \geq x\}$ is continuous at $a_0 = \xi_{\sup}(\alpha)$, then it follows from Corollary 3.1 that $\text{Cr}\{\xi \geq a_0\} = \alpha$. The proof of the theorem is complete.

Theorem 3.5 gives a sufficient condition under which the equality $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha$ holds. In what follows, we will establish a sufficient and necessary condition for the equality $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha$.

Since $\text{Cr}\{\xi \geq x\}$ is a non-increasing function on $\Re$, its discontinuous points are at most countable. We denote $E = \{x_1, x_2, \cdots\}$ the set of all discontinuous points of $\text{Cr}\{\xi \geq x\}$. For every $x_i \in E$, denote $a_i = \lim_{x\to x_i-} \text{Cr}\{\xi \geq x\}$, $c_i = \text{Cr}\{\xi \geq x_i\}$, $b_i = \lim_{x\to x_i+} \text{Cr}\{\xi \geq x\}$. Then each $x_i \in E$ must satisfy only one of the following three conditions:

$C1$**:** $\text{Cr}\{\xi \geq x\}$ is left continuous at x_i, i.e., $a_i = c_i > b_i$.

$C2$**:** $\text{Cr}\{\xi \geq x\}$ is right continuous at x_i, i.e., $a_i > c_i = b_i$.

$C3$**:** $\text{Cr}\{\xi \geq x\}$ is neither left continuous nor right continuous at x_i, i.e., $a_i > c_i > b_i$.

Furthermore, we decompose conditions $C1$, $C2$ into the following cases:

$C1.a$**:** $a_i = c_i > b_i$, and there exists $\delta > 0$ such that for any x with $0 < x - x_i < \delta$, we have $\text{Cr}\{\xi \geq x\} = b_i$.

$C1.b$**:** $a_i = c_i > b_i$, and $\text{Cr}\{\xi \geq x\} < b_i$ for all $x > x_i$.

$C3.a$**:** $a_i > c_i > b_i$, and there exists $\delta > 0$ such that for any x with $0 < x - x_i < \delta$, we have $\text{Cr}\{\xi \geq x\} = b_i$.

$C3.b$**:** $a_i > c_i > b_i$, and $\text{Cr}\{\xi \geq x\} < b_i$ for all $x > x_i$.

Proposition 3.2 *Let ξ be a fuzzy variable, and $x_i \in E$. Then we have*

(i) $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} > \alpha$ *for any* $\alpha \in (b_i, a_i)$ *provided* x_i *satisfies* $C1.a$*;*

(ii) $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} > \alpha$ *for any* $\alpha \in [b_i, a_i)$ *provided* x_i *satisfies* $C1.b$*;*

(iii) $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} < \alpha$ *for any* $\alpha \in (b_i, a_i]$ *provided* x_i *satisfies* $C2$*;*

(iv) $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} > \alpha$ *for any* $\alpha \in (b_i, c_i)$*, and* $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} < \alpha$ *for any* $\alpha \in (c_i, a_i]$ *provided* x_i *satisfies* $C3.a$*;*

(v) $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} > \alpha$ *for any* $\alpha \in [b_i, c_i)$*, and* $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} < \alpha$ *for any* $\alpha \in (c_i, a_i]$ *provided* x_i *satisfies* $C3.b$*.*

Proof. We only prove (i), (ii), (iii), the rest can be proved in a similar way.

(i) Since $a_i = \text{Cr}\{\xi \geq x_i\}$, it suffices to prove for any $\alpha \in (b_i, a_i)$,

$$x_i = \xi_{\sup}(\alpha) = \sup\{x \mid \text{Cr}\{\xi \geq x\} \geq \alpha\}.$$

Noting that for any $x > x_i$, $\text{Cr}\{\xi \geq x\} \leq \lim_{x\to x_i+} \text{Cr}\{\xi \geq x\} = b_i < \alpha$, which is equivalent to if $\text{Cr}\{\xi \geq x\} \geq \alpha$, then $x \leq x_i$. Since $\text{Cr}\{\xi \geq x_i\} = a_i > \alpha$, we can obtain

$$x_i = \xi_{\sup}(\alpha) = \sup\{x \mid \text{Cr}\{\xi \geq x\} \geq \alpha\},$$

which implies for any $\alpha \in (b_i, a_i)$, $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = a_i > \alpha$.

(ii) By the proof of (1), we only need to prove

$$\text{Cr}\{\xi \geq \xi_{\sup}(b_i)\} > b_i.$$

Since x_i satisfies $C1.b$, for any $x > x_i$, we have $\mathrm{Cr}\{\xi \geq x\} < b_i$. Noting that $\mathrm{Cr}\{\xi \geq x_i\} > b_i$, we have

$$x_i = \xi_{\sup}(b_i) = \sup\{x \mid \mathrm{Cr}\{\xi \geq x\} \geq b_i\},$$

which implies $\mathrm{Cr}\{\xi \geq \xi_{\sup}(b_i)\} > b_i$.

(iii) Since $\mathrm{Cr}\{\xi \geq x_i\} = b_i$, it suffices to prove for any $\alpha \in (b_i, a_i]$,

$$x_i = \xi_{\sup}(\alpha) = \sup\{x \mid \mathrm{Cr}\{\xi \geq x\} \geq \alpha\}.$$

On the one hand, for any $x > x_i$, we have

$$\mathrm{Cr}\{\xi \geq x\} \leq \lim_{x \to x_i+} \mathrm{Cr}\{\xi \geq x\} = b_i < \alpha.$$

That is if $\mathrm{Cr}\{\xi \geq x\} \geq \alpha$, then $x \leq x_i$.

On the other hand, since $\lim_{x \to x_i-} \mathrm{Cr}\{\xi \geq x\} = a_i \geq \alpha$, and $\mathrm{Cr}\{\xi \geq x\}$ is non-increasing, for any $\varepsilon > 0$, there is an x^* with $x^* + \varepsilon > x_i > x^*$ such that

$$\mathrm{Cr}\{\xi \geq x^*\} \geq \lim_{x \to x_i-} \mathrm{Cr}\{\xi \geq x\} \geq \alpha.$$

Consequently, by the definition of $\xi_{\sup}(\alpha)$, we obtain

$$x_i = \xi_{\sup}(\alpha) \text{ and } \mathrm{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = b_i < \alpha.$$

The proof of the proposition is complete.

The following example shows when x_i satisfies $C1.a$, and $\alpha = b_i$, $\mathrm{Cr}\{\xi \geq \xi_{\sup}(b_i)\} > \alpha = b_i$ does not hold.

Example 3.1 *Let ξ be a fuzzy variable with possibility distribution*

$$\mu(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } 0 < x \leq 1 \\ 0.6, & \text{if } 1 < x \leq 2 \\ 0, & \text{if } x > 2. \end{cases}$$

Then, we have

$$\mathrm{Cr}\{\xi \geq x\} = \begin{cases} 1, & \text{if } x \leq 0 \\ 0.5, & \text{if } 0 < x \leq 1 \\ 0.3, & \text{if } 1 < x \geq 2 \\ 0, & \text{if } 2 < x. \end{cases}$$

$x_i = 1$ is a discontinuous point of $\mathrm{Cr}\{\xi \geq x\}$, i.e., $1 \in E$, and it satisfies the condition $C1.a$. In fact, $a_i = 0.5 = \mathrm{Cr}\{\xi \geq 1\} > 0.3 = b_i = \lim_{x \to 1+} \mathrm{Cr}\{\xi \geq x\}$, and there exists $0 < \delta < 1$ such that for all x with $0 < x - 1 < \delta$, $\mathrm{Cr}\{\xi \geq x\} = 0.3 = b_i$. However, $\xi_{\sup}(b_i) = \xi_{\sup}(0.3) = 2$, and $\mathrm{Cr}\{\xi \geq 2\} = 0.3 = b_i$.

Proposition 3.3 *Let ξ be a fuzzy variable and $\alpha \in (0, 1]$, $a = \xi_{\sup}(\alpha)$. Then we have*

$$\lim_{x \to a+} \mathrm{Cr}\{\xi \geq x\} \leq \alpha \leq \lim_{x \to a-} \mathrm{Cr}\{\xi \geq x\}.$$

Proof. On one hand, noting that $a = \xi_{\sup}(\alpha) = \sup\{x \mid \text{Cr}\{\xi \geq x\} \geq \alpha\}$, and $\text{Cr}\{\xi \geq x\}$ is decreasing, we can obtain $\lim\limits_{x \to a+} \text{Cr}\{\xi \geq x\} \leq \alpha$.

On the other hand, we use a proof by contradiction, suppose $\alpha > \lim\limits_{x \to a-} \text{Cr}\{\xi \geq x\}$, then there is an $x_0 < a$ such that $\text{Cr}\{\xi \geq x_0\} < \alpha$. However, noting that $a = \sup\{x \mid \text{Cr}\{\xi \geq x\} \geq \alpha\}$ and $\text{Cr}\{\xi \geq x\}$ is decreasing, we deduce for any $x < a$, we have $\text{Cr}\{\xi \geq x\} \geq \alpha$, which leads to a contradiction. Therefore, $\alpha \leq \lim\limits_{x \to a-} \text{Cr}\{\xi \geq x\}$. The proof is complete.

For any $x_i \in E$, we denote

$$I\langle x_i \rangle = \begin{cases} (b_i, a_i), & \text{if } x_i \text{ satisfies } C1.a \\ [b_i, a_i), & \text{if } x_i \text{ satisfies } C1.b \\ (b_i, a_i], & \text{if } x_i \text{ satisfies } C2 \\ (b_i, c_i) \cup (c_i, a_i], & \text{if } x_i \text{ satisfies } C3.a \\ [b_i, c_i) \cup (c_i, a_i], & \text{if } x_i \text{ satisfies } C3.b. \end{cases}$$

The sufficient and necessary condition for the equality $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha$ is discussed in the following theorem.

Theorem 3.6 *Let ξ be a fuzzy variable, and E the set of all discontinuous points of* $\text{Cr}\{\xi \geq x\}$. *Then for any* $\alpha \in (0, 1]$, $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha$ *if and only if* $\alpha \notin \bigcup\limits_{x_i \in E} I\langle x_i \rangle$.

Proof. *Sufficiency:* If $\alpha \notin \bigcup\limits_{x_i \in E} I\langle x_i \rangle$, α must satisfy only one of the following six conditions:

(i) There is an $x_i \in E$, x_i satisfies $C1.a$ such that $\alpha = b_i$.

(ii) There is an $x_i \in E$, x_i satisfies $C1$ such that $\alpha = a_i$.

(iii) There is an $x_i \in E$, x_i satisfies $C2$ such that $\alpha = b_i$.

(iv) There is an $x_i \in E$, x_i satisfies $C3.a$ such that $\alpha = b_i$.

(v) There is an $x_i \in E$, x_i satisfies $C3$ such that $\alpha = c_i$.

(vi) $\xi_{\sup}(\alpha) \notin E$.

In fact, if $\alpha \notin \bigcup\limits_{x_i \in E} I\langle x_i \rangle$ and does not satisfy any one of conditions (i), (ii), (iii), (iv), (v), then $\alpha \notin \bigcup\limits_{x_i \in E} [b_i, a_i]$, which follows that $\xi_{\sup}(\alpha) \notin E$. Otherwise, if $\xi_{\sup}(\alpha) = x_j \in E$, by Proposition 3.3, we deduce $b_j \leq \alpha \leq a_j$ which leads to a contradiction.

We only need to prove $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha$ under the conditions (i), (ii), (iii), (iv), (v). By Corollary 3.1, $\text{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha$ is obviously valid under condition (vi).

(i) Since $x_i \in E$ satisfies $C1.a$ and $\alpha = b_i$, noting that $\lim\limits_{x \to \infty} \text{Cr}\{\xi \geq x\} = 0$, there is a $\delta_0 > 0$, for any x with $0 < x - x_i < \delta_0$, and any x with $x > \delta_0 + x_i$, one has $\text{Cr}\{\xi \geq x\} = b_i$ and $\text{Cr}\{\xi \geq x\} < b_i$ respectively. Denote $x_j = \delta_0 + x_i$. Then

$$b_i = \lim_{x \to x_j -} \text{Cr}\{\xi \geq x\} = \text{Cr}\{\xi \geq x_j\}.$$

Otherwise, if $b_i = \lim\limits_{x \to x_j -} \text{Cr}\{\xi \geq x\} > \text{Cr}\{\xi \geq x_j\}$, then x_j satisfies $C2$ or $C3$ such that $\alpha = b_i = a_j \in I\langle x_j \rangle$, which is a contradiction with $\alpha \notin \bigcup\limits_{x_i \in E} I\langle x_i \rangle$. Since for any x with $x > x_j$, we have $\text{Cr}\{\xi \geq x\} < b_i = \alpha$. It follows from $\text{Cr}\{\xi \geq x_j\} = b_i$ that

$$x_j = \xi_{\sup}(b_i) \text{ and } \text{Cr}\{\xi \geq \xi_{\sup}(b_i)\} = b_i.$$

(ii) Since $x_i \in E$ satisfies $C1$, and $\alpha = a_i$, one has

$$a_i = \mathrm{Cr}\{\xi \geq x_i\} > \lim_{x \to x_i+} \mathrm{Cr}\{\xi \geq x\} = b_i.$$

To prove (ii), it suffices to prove $x_i = \xi_{\sup}(a_i) = \sup\{x \mid \mathrm{Cr}\{\xi \geq x\} \geq a_i\}$. Since for any x with $x > x_i$, one has $\mathrm{Cr}\{\xi \geq x\} \leq \lim_{x \to x_i+} \mathrm{Cr}\{\xi \geq x\} = b_i < a_i$. Noting that $\mathrm{Cr}\{\xi \geq x_i\} = a_i$, it follows that

$$x_i = \xi_{\sup}(a_i) \text{ and } \mathrm{Cr}\{\xi \geq \xi_{\sup}(a_i)\} = a_i.$$

(iii) If $x_i \in E$ satisfies $C2$, and $\alpha = b_i$, we have

$$a_i > \mathrm{Cr}\{\xi \geq x_i\} = \lim_{x \to x_i+} \mathrm{Cr}\{\xi \geq x\} = b_i.$$

On one hand, if for any $x > x_i$, $\mathrm{Cr}\{\xi \geq x\} < b_i$, then it follows from $\mathrm{Cr}\{\xi \geq x_i\} = b_i$ that

$$x_i = \xi_{\sup}(b_i) \text{ and } \mathrm{Cr}\{\xi \geq \xi_{\sup}(b_i)\} = b_i.$$

On the other hand, if there is a $\delta > 0$, for any x with $0 < x - x_i < \delta$, one has $\mathrm{Cr}\{\xi \geq x\} = b_i$. Noting that $\lim_{x \to \infty} \mathrm{Cr}\{\xi \geq x\} = 0$, there is a δ_0 such that for any x with $0 < x - x_i < \delta_0$, $\mathrm{Cr}\{\xi \geq x\} = b_i$, and for any x with $x > x_i + \delta_0$, $\mathrm{Cr}\{\xi \geq x\} < b_i$ respectively. Denote $x_j = x_i + \delta_0$. Then we can get

$$b_i = \lim_{x \to x_j-} \mathrm{Cr}\{\xi \geq x\} = \mathrm{Cr}\{\xi \geq x_j\}.$$

Otherwise, if $b_i = \lim_{x \to x_j-} \mathrm{Cr}\{\xi \geq x\} > \mathrm{Cr}\{\xi \geq x_j\}$, then x_j satisfies $C2$ or $C3$ such that $\alpha = b_i = a_j \in I\langle x_j \rangle$, which is a contradiction with $\alpha \notin \bigcup_{x_i \in E} I\langle x_i \rangle$.

The rest of the proof for (iii) can be completed by the same method with the proof in (i). Thus, we can obtain $\mathrm{Cr}\{\xi \geq \xi_{\sup}(b_i)\} = b_i$.

(iv) The proof of (iv) is similar to the proof in (i).

(v) Since $x_i \in E$ satisfies $C3$ and $\alpha = c_i = \mathrm{Cr}\{\xi \geq x_i\}$, we have

$$a_i > c_i = \mathrm{Cr}\{\xi \geq x_i\} > \lim_{x \to x_i+} \mathrm{Cr}\{\xi \geq x\} = b_i.$$

Noting that for any $x > x_i$, $\mathrm{Cr}\{\xi \geq x\} \leq b_i < c_i$, by $\mathrm{Cr}\{\xi \geq x_i\} = c_i$, we can get

$$x_i = \xi_{\sup}(c_i) = \{x \mid \mathrm{Cr}\{\xi \geq x\} \geq c_i\}.$$

Thus $\mathrm{Cr}\{\xi \geq \xi_{\sup}(c_i)\} = c_i = \alpha$.

Necessity: It follows from Proposition 3.2 that for any $\alpha \in \bigcup_{x_i \in E} I\langle x_i \rangle$, we have

$$\mathrm{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} \neq \alpha,$$

which proves the necessity of the theorem.

Corollary 3.2 *Let ξ be a fuzzy variable, E the set of all discontinuous points of* $\mathrm{Cr}\{\xi \geq x\}$. *Then for any* $\alpha \in (0,1] \setminus \bigcup_{x_i \in E} [b_i, a_i]$, *we have* $\mathrm{Cr}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha$.

3.3 On the Continuity of $\xi_{\sup}(\alpha)$

It is known from [7, Theorem 3.30] that for any $\alpha \in (0,1]$, $\xi_{\sup}(\alpha)$ is a left continuous function of α. The following theorem gives a sufficient and necessary condition under which $\xi_{\sup}(\alpha)$ is continuous with respect to α.

Theorem 3.7 *Let ξ be a fuzzy variable. Then for any $\alpha \in (0,1]$, $\xi_{\sup}(\alpha)$ is continuous at α if and only if there is at most one value x such that* $\mathrm{Cr}\{\xi \geq x\} = \alpha$.

Proof. *Sufficiency:* Since for any $\alpha \in (0,1]$, $\xi_{\sup}(\alpha)$ is a left continuous function of α, we just need to prove $\xi_{\sup}(\alpha)$ is right continuous with respect to α.

It is easy to know that $\xi_{\sup}(\alpha)$ is a decreasing function of α. Let $\{\alpha_i\}$ be an arbitrary sequence of positive numbers such that $\alpha_i \downarrow \alpha$. Then $\{\xi_{\sup}(\alpha_i)\}$ is an increasing sequence. If the limit is equal to $\xi_{\sup}(\alpha)$, then the right continuity is proved. Otherwise, we have $\lim_{i\to\infty} \xi_{\sup}(\alpha_i) < \xi_{\sup}(\alpha)$. Let z_1, z_2 be two points such that

$$\lim_{i\to\infty} \xi_{\sup}(\alpha_i) < z_1 < z_2 < \xi_{\sup}(\alpha).$$

It is clear that $\xi_{\sup}(\alpha_i) < z_1 < z_2 < \xi_{\sup}(\alpha)$ for all i. Then $\mathrm{Cr}\{\xi \geq z_1\} < \alpha_i, \mathrm{Cr}\{\xi \geq z_2\} < \alpha_i$ for all i. Hence,

$$\mathrm{Cr}\{\xi \geq z_1\} \leq \alpha, \ \ \mathrm{Cr}\{\xi \geq z_2\} \leq \alpha.$$

If $\mathrm{Cr}\{\xi \geq z_i\} < \alpha$, $i = 1$ or 2, then $z_i \geq \xi_{\sup}(\alpha)$, $i = 1$ or 2, which is a contradiction with $z_1 < z_2 < \xi_{\sup}(\alpha)$. As a consequence, $\mathrm{Cr}\{\xi \geq z_i\} = \alpha$, $i = 1, 2$, which is also a contradiction with that there is at most one value x such that $\mathrm{Cr}\{\xi \geq x\} = \alpha$. The above contradictions prove that

$$\lim_{i\to\infty} \xi_{\sup}(\alpha_i) = \xi_{\sup}(\alpha).$$

Combining with the left continuity of $\xi_{\sup}(\alpha)$, $\xi_{\sup}(\alpha)$ is continuous with respect to α.

Necessity: Suppose there are two points t_1, t_2 such that $t_1 < t_2$ and $\mathrm{Cr}\{\xi \geq t_1\} = \mathrm{Cr}\{\xi \geq t_2\} = \alpha$, then we get $\mathrm{Cr}\{\xi \geq t\} = \alpha$ for any $t_1 \leq t \leq t_2$, and

$$\xi_{\sup}(\alpha) = \sup\{x \mid \mathrm{Cr}\{\xi \geq x\} \geq \alpha\} \geq t_2.$$

For any $\delta > 0$, if we take an α_0 with $0 < \alpha_0 - \alpha < \delta$, then for any x with $\mathrm{Cr}\{\xi \geq x\} \geq \alpha_0 > \alpha$, we have $x < t_1$, which follows from the definition of $\xi_{\sup}(\alpha_0)$ that

$$\xi_{\sup}(\alpha_0) = \sup\{x \mid \mathrm{Cr}\{\xi \geq x\} \geq \alpha_0\} \leq t_1.$$

Thus, we obtain $\xi_{\sup}(\alpha_0) \leq t_1 < t_2 \leq \xi_{\sup}(\alpha)$, which is a contradiction with the continuity of $\xi_{\sup}(\alpha)$ at α. The necessity is proved.

4 The Properties of Pessimistic Value Functions

In this section, we discuss the properties of the credibility pessimistic value functions, which can be proved similarly to those of the credibility optimistic value functions in Section 3. So, we just provide the results and omit the proofs.

Proposition 4.1 *Let ξ be a fuzzy variable. Then for any $\alpha \in (0,1]$, $\xi_{\inf}(\alpha) > -\infty$ if and only if* $\lim_{x\to-\infty} \mathrm{Cr}\{\xi \leq x\} = 0$.

We assume in this section that fuzzy variable ξ satisfies $\lim_{x \to -\infty} \text{Cr}\{\xi \leq x\} = 0$. Thus, it follows from Proposition 4.1 that for any $\alpha \in (0, 1]$, $\xi_{\inf}(\alpha) > -\infty$.

Theorem 4.1 *Let ξ be a fuzzy variable. Then we have*

(i) $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} \geq \alpha$ *provided that* $\text{Cr}\{\xi \leq x\}$ *is right continuous at* $\xi_{\inf}(\alpha)$*;*

(ii) $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} \leq \alpha$ *provided that* $\text{Cr}\{\xi \leq x\}$ *is left continuous at* $\xi_{\inf}(\alpha)$*.*

Theorem 4.2 *Let ξ be a fuzzy variable.*

(i) $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} \geq \alpha$ *for every* $\alpha \in (0, 1]$ *if and only if* $\text{Cr}\{\xi \leq x\}$ *is right continuous;*

(ii) $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} \leq \alpha$ *for every* $\alpha \in (0, 1]$ *if and only if* $\text{Cr}\{\xi \leq x\}$ *is left continuous.*

Theorem 4.3 *Let ξ be a fuzzy variable. Then* $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} \geq \alpha$ *for every* $\alpha \in (0, 1]$ *if and only if for any* $x \in \Re$, $\limsup_{t \to x+} \mu(t) \leq \text{Pos}\{\xi \leq x\}$.

Corollary 4.1 *If ξ is a fuzzy variable, and* $\text{Cr}\{\xi \leq x\}$ *is continuous at* $\xi_{\inf}(\alpha)$*, then* $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} = \alpha$*.*

Theorem 4.4 *Let ξ be a lower semicontinuous fuzzy variable with possibility distribution $\mu(x)$, and $a_0 = \xi_{\inf}(\alpha)$. If*

$$\limsup_{x \to a_0+} \mu(x) \leq \text{Pos}\{\xi \leq a_0\}, \text{ and } \limsup_{x \to a_0-} \mu(x) \leq \text{Pos}\{\xi \geq a_0\},$$

then $\text{Cr}\{\xi \leq a_0\} = \alpha$*.*

Let $E = \{x_1, x_2, \cdots\}$ be the set of all discontinuous points of $\text{Cr}\{\xi \leq x\}$. For each $x_i \in E$, denote $a_i = \lim_{x \to x_i-} \text{Cr}\{\xi \leq x\}$, $c_i = \text{Cr}\{\xi \leq x_i\}$, $b_i = \lim_{x \to x_i+} \text{Cr}\{\xi \leq x\}$. Then each point $x_i \in E$ must satisfy only one of the following five conditions:

$C1.a$**:** $a_i < c_i = b_i$, and there exists $\delta > 0$ such that for any x with $0 < x_i - x < \delta$, we have $\text{Cr}\{\xi \leq x\} = a_i$.

$C1.b$**:** $a_i < c_i = b_i$, and $\text{Cr}\{\xi \leq x\} < a_i$ for all $x < x_i$.

$C2$**:** $a_i = c_i < b_i$.

$C3.a$**:** $a_i < c_i < b_i$, and there exists $\delta > 0$ such that for any x with $0 < x_i - x < \delta$, we have $\text{Cr}\{\xi \leq x\} = a_i$.

$C3.b$**:** $a_i < c_i < b_i$, and $\text{Cr}\{\xi \leq x\} < a_i$ for all $x_i < x$.

Proposition 4.2 *Let ξ be a fuzzy variable, and $x_i \in E$. Then we have*

(i) $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} > \alpha$ *for any* $\alpha \in (a_i, b_i)$ *provided x_i satisfies $C1.a$.*

(ii) $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} > \alpha$ *for any* $\alpha \in [a_i, b_i)$ *provided x_i satisfies $C1.b$.*

(iii) $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} < \alpha$ *for any* $\alpha \in (a_i, b_i]$ *provided x_i satisfies $C2$.*

(iv) $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} > \alpha$ *for any* $\alpha \in (a_i, c_i)$*, and* $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} < \alpha$ *for any* $\alpha \in (c_i, b_i]$ *provided x_i satisfies $C3.a$.*

(v) $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} > \alpha$ *for any* $\alpha \in [a_i, c_i)$*, and* $\text{Cr}\{\xi \leq \xi_{\inf}(\alpha)\} < \alpha$ *for any* $\alpha \in (c_i, b_i]$ *provided x_i satisfies $C3.b$.*

Proposition 4.3 *Let ξ be a fuzzy variable and $\alpha \in (0,1]$, $a = \xi_{\inf}(\alpha)$. Then we have*

$$\lim_{x \to a-} \mathrm{Cr}\{\xi \le x\} \le \alpha \le \lim_{x \to a+} \mathrm{Cr}\{\xi \le x\}.$$

For any $x_i \in E$, we denote

$$I\langle x_i \rangle = \begin{cases} (a_i, b_i), & \text{if } x_i \text{ satisfies } C1.a \\ [a_i, b_i), & \text{if } x_i \text{ satisfies } C1.b \\ (a_i, b_i], & \text{if } x_i \text{ satisfies } C2 \\ (a_i, c_i) \cup (c_i, b_i], & \text{if } x_i \text{ satisfies } C3.a \\ [a_i, c_i) \cup (c_i, b_i], & \text{if } x_i \text{ satisfies } C3.b. \end{cases}$$

The sufficient and necessary condition for the equality $\mathrm{Cr}\{\xi \le \xi_{\inf}(\alpha)\} = \alpha$ is given by the following theorem.

Theorem 4.5 *Let ξ be a fuzzy variable, and E the set of all discontinuous points of* $\mathrm{Cr}\{\xi \le x\}$. *Then for any* $\alpha \in (0,1]$, $\mathrm{Cr}\{\xi \le \xi_{\inf}(\alpha)\} = \alpha$ *if and only if* $\alpha \notin \bigcup_{x_i \in E} I\langle x_i \rangle$.

Corollary 4.2 *Let ξ be a fuzzy variable, E the set of all discontinuous points of* $\mathrm{Cr}\{\xi \le x\}$. *Then for any* $\alpha \in (0,1] \setminus \bigcup_{x_i \in E} [a_i, b_i]$, *we have* $\mathrm{Cr}\{\xi \le \xi_{\inf}(\alpha)\} = \alpha$.

By [7, Theorem 3.30], we have known that for any $\alpha \in (0,1]$, $\xi_{\inf}(\alpha)$ is a left continuous function of α. The following theorem gives the sufficient and necessary condition for the continuity of $\xi_{\inf}(\alpha)$ with respect to α.

Theorem 4.6 *Let ξ be a fuzzy variable. Then for any $\alpha \in (0,1]$, $\xi_{\inf}(\alpha)$ is continuous at α if and only if there is at most one value x such that* $\mathrm{Cr}\{\xi \le x\} = \alpha$.

5 Conclusions

In this paper, based on the continuity of credibility functions $\mathrm{Cr}\{\xi \ge x\}$ and $\mathrm{Cr}\{\xi \le x\}$, we discussed the properties of the credibility critical value functions, and obtained the following major new results:

(i) The sufficient and necessary conditions under which the inequalities $\mathrm{Cr}\{\xi \ge \xi_{\sup}(\alpha)\} \ge \alpha$ and $\mathrm{Cr}\{\xi \le \xi_{\inf}(\alpha)\} \ge \alpha$ hold for any $\alpha \in (0,1]$ were established.

(ii) The sufficient conditions under which the inequalities $\mathrm{Cr}\{\xi \ge \xi_{\sup}(\alpha)\} \le \alpha$ and $\mathrm{Cr}\{\xi \le \xi_{\inf}(\alpha)\} \le \alpha$ hold for any $\alpha \in (0,1]$ were given.

(iii) The sufficient and necessary conditions for the equalities $\mathrm{Cr}\{\xi \ge \xi_{\sup}(\alpha)\} = \alpha$ and $\mathrm{Cr}\{\xi \le \xi_{\inf}(\alpha)\} = \alpha$ were discussed.

(iv) The sufficient and necessary conditions for the continuity of $\xi_{\sup}(\alpha)$ and $\xi_{\inf}(\alpha)$ with respect to α were provided.

Acknowledgments

This work was supported by National Natural Science Foundation of China under Grant 70571021, and Natural Science Foundation of Hebei Province under Grant A2005000087.

References

[1] Dubois, D., and Prade, H.,*Possibility Theory,* New York: Plenum, 1988.

[2] Inuiguchi, M., Ichihashi, H., and Kume, Y., Modality constrained programming problems: a unified approach to fuzzy mathematical programming problems in the setting of possibility theory, *Information Sciences*, Vol. 67, 93–126, 1993.

[3] Klir, G. J., On fuzzy-set interpretation of possibility theory, *Fuzzy Sets Syst.,* Vol. 108, 263–273, 1999.

[4] Kall, P., and Wallace, S. W., *Stochastic Programming*, Chichester: Wiley, 1994.

[5] Liu, Y.-K., and Wang, S. M., The sufficient and necessary conditions for the continuity of credibility functions, *Technical Report*, 2005.

[6] Liu, B., and Liu, Y.-K. Expected value of fuzzy variable and fuzzy expected value models, *IEEE Trans. Fuzzy Syst.*, Vol. 10, 445–450, 2002.

[7] Liu, B. *Uncertainty Theory: An Introduction to Its Axiomatic Foundations*, Springer-Verlag, Berlin, 2004.

[8] Liu, B. *Theory and Practice of Uncertain Programming*, Physica-Verlag, Heidelberg, 2002.

[9] Liu, B., Toward fuzzy optimization without mathematical ambiguity, *Fuzzy Optimization and Decision Making,* Vol. 1, 43–63, 2002.

[10] Lai, Y.-J., and Hwang, C.-L., *Fuzzy Mathematical Programming: Methods and Applications*, Springer-Verlag, New York, 1992.

[11] Mukherjea, A., and Pothoven, K. *Real and Functional Analysis*, New York: Plenum, 1984.

[12] Nahmias, S. Fuzzy variables, *Fuzzy Sets Syst.*, Vol. 1, 97–101, 1978.

[13] Royden, H. L., *Real Analysis*, New York: Macmillan, 1963.

[14] Sakawa, M., *Fuzzy Sets and Interactive Multiobjective Optimization*, New York: Plenum, 1993.

[15] Wang, P., Fuzzy contactability and fuzzy variables, *Fuzzy Sets Syst.,* Vol 8, 81–92, 1982.

[16] Yager, R. R., On the specificity of a possibility distribution, *Fuzzy Sets Syst.* Vol. 50, 279–292, 1992.

[17] Zadeh, L. A., Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets Syst.*, Vol. 1, 3–28, 1978.