Some Theoretical Aspects of the Critical Values of Birandom Variable

Jin Peng[1*],   Xiande Zhao[2]

[1] Department of Mathematics, Huanggang Normal University, Huanggang 438000, Hubei, China
[2] Department of Decision Sciences and Managerial Economics, Faculty of Business Administration
The Chinese University of Hong Kong, Shatin, Hong Kong

Abstract.   There are two types of critical values for birandom variable: the optimistic and pessimistic values. In this paper, some theoretical aspects of optimistic and pessimistic values of birandom variable are investigated. Based on the results, some properties of birandom chance-constrained programming models can be further discussed.

Keywords: Birandom variable, optimistic value, pessimistic value, birandom programming

1   Introduction

As a general mathematical description for the stochastic phenomenon with incomplete statistical information, birandom variable is firstly introduced by Liu [9] and studied by Peng & Liu [17][18], Yang & Liu [20]. Birandom variable is defined as a mapping with some kind of measurability from a probability space to a collection of random variables. Naturally, birandom variable is an extension of conventional random variable and it plays an important role in birandom decision system as good as a random variable does in probability theory. Similar to the case of random variable, we are concerned with the mathematical properties of the optimistic and pessimistic values of birandom variables [10].

In probability theory and statistics, the quantile function (or fractile function) of a random variable is widely studied by many researchers. In fact, the quantile function is exactly the pessimistic value of a random variable in optimization. There are internal relationships between the optimistic value and pessimistic value of a random variable. For more detail about the quantile functions of random variables, we may consult the references [2][3][5][6][7][10] [15][19].

In birandom theory, the optimistic and pessimistic values of a birandom variable are also very important concepts, which play a key role in birandom chance-constrained programming. Actually, they act as a class of optimization objects in birandom environments.

In this paper, some aspects of optimistic and pessimistic values of birandom variable are investigated. Based on the results, some properties of birandom chance-constrained programming models can be further discussed and expected to use in sensitivity analysis of birandom programming.

The remaining sections of the paper are organized as follows. In the next section, the basic concepts related to birandom variable are introduced. In Section 3 is devoted to studying some properties of the optimistic and

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* Corresponding author. Tel: +86-713-8621602, Fax: +86-713-8621601.
E-mail address: pengjin01@tsinghua.org.cn (J. Peng).

pessimistic values of birandom variables. Section 4 illustrates one of the properties to birandom optimization. Finally, Section 5 concludes this paper with a summary.

2 Preliminary Concepts

As the extentions of random variable from different angles, Banach-valued random variables, fuzzy random variables and random fuzzy variables have been studied by some researchers such as Araujo[1], Cuesta & Matran[4], Merlevéde[16], Liu [11][12], and Liu & Liu [13][14].

Birandom variable is an extension of conventional random variable from a new angle of view. It is a kind of mathematical tool to describe a birandom phenomenon. Peng and Liu [17] studied birandom variable comprehensively, and defined the chance measure of birandom event and the critical values of birandom variable.

Definition 1 *Let $(\Omega, A, \Pr)$ be a probability space. A birandom variable ξ is a mapping from a probability space $(\Omega, A, \Pr)$ to a collection of random variables S such that $\Pr\{\xi(\omega) \in B\}$ is a measurable function with respect to ω for any Borel subset B of the real line $\Re$.*

Definition 2 *An n-dimensional birandom vector $\boldsymbol{\xi}$ is a mapping from the probability space $(\Omega, A, \Pr)$ to a collection of n-dimensional random vector such that $\Pr\{\boldsymbol{\xi}(\omega) \in \boldsymbol{B}\}$ is a measurable function with respect to ω for any Borel subset $\boldsymbol{B}$ of the real space $\Re^n$.*

Definition 3 *Let ξ be a birandom variable on $(\Omega, A, \Pr)$, and B a Borel set of $\Re$. Then the chance of birandom event characterized by $\xi \in B$ is a function from $(0, 1]$ to $[0, 1]$, defined as*

$$\mathrm{Ch}\left\{\xi \in B\right\}(\alpha) = \sup_{\Pr\{A\} \geq \alpha} \inf_{\omega \in A} \Pr\left\{\xi(\omega) \in B\right\}.$$

Remark: Equivalently, the chance measure may be written as

$$\mathrm{Ch}\left\{\xi \in B\right\}(\alpha) = \sup\left\{\beta \mid \Pr\left\{\omega \in \Omega \mid \Pr\left\{\xi(\omega) \in B\right\} \geq \beta\right\} \geq \alpha\right\}.$$

There are two types of critical values for birandom variable: the optimistic and pessimistic values. Below, we give the definitions of optimistic value and pessimistic value of birandom variable.

Definition 4 *Let ξ be a birandom variable and $\gamma, \delta \in (0, 1]$. Then*

$$\xi_{\sup}(\gamma, \delta) = \sup\left\{x | \mathrm{Ch}\{\xi \geq x\}(\gamma) \geq \delta\right\} \tag{1}$$

is called the (γ, δ)-optimistic value to ξ, and

$$\xi_{\inf}(\gamma, \delta) = \inf\left\{x | \mathrm{Ch}\{\xi \leq x\}(\gamma) \geq \delta\right\} \tag{2}$$

is called the (γ, δ)-pessimistic value to ξ.

3 Propositions

In this section, we proceed to prove some theorems to show some properties of the optimistic and pessimistic values of birandom variables.

Theorem 1 *Let ξ be a birandom variable and $\gamma, \delta \in (0, 1]$ be given. Assume that $\xi_{\sup}(\gamma, \delta)$ is the (γ, δ)-optimistic value and $\xi_{\inf}(\gamma, \delta)$ is the (γ, δ)-pessimistic value to ξ. Then we have*
(a) $\mathrm{Ch}\{\xi \leq \xi_{\inf}(\gamma, \delta)\}(\gamma) \geq \delta$;
(b) $\mathrm{Ch}\{\xi \geq \xi_{\sup}(\gamma, \delta)\}(\gamma) \geq \delta$.

Proof. It follows from the definition of (γ, δ)-pessimistic value that there exists a decreasing sequence $\{x_n\}$ such that $\mathrm{Ch}\{\xi \leq x_n\}(\gamma) \geq \delta$ and $x_n \downarrow \xi_{\inf}(\gamma, \delta)$ as $n \to \infty$. Since $\mathrm{Ch}\{\xi \leq x\}(\gamma)$ is a right-continuous function of x, the inequality $\mathrm{Ch}\{\xi \leq \xi_{\inf}(\gamma, \delta)\}(\gamma) \geq \delta$ holds.

Similarly, there exists an increasing sequence $\{x_n\}$ such that $\mathrm{Ch}\{\xi \geq x_n\}(\gamma) \geq \delta$ and $x_n \uparrow \xi_{\sup}(\gamma, \delta)$ as $n \to \infty$. Since $\mathrm{Ch}\{\xi \geq x\}(\gamma)$ is a left-continuous function of x, the inequality $\mathrm{Ch}\{\xi \geq \xi_{\sup}(\gamma, \delta)\}(\gamma) \geq \delta$ holds. The theorem is proved.

Theorem 2 *Let ξ be a birandom variable, and $\gamma, \delta \in (0, 1]$. Then we have*
(a) if $\lambda \geq 0$, then

$$(\lambda\xi)_{\sup}(\gamma, \delta) = \lambda\xi_{\sup}(\gamma, \delta), \tag{3}$$

and

$$(\lambda\xi)_{\inf}(\gamma, \delta) = \lambda\xi_{\inf}(\gamma, \delta); \tag{4}$$

(b) if $\lambda < 0$, then

$$(\lambda\xi)_{\sup}(\gamma, \delta) = \lambda\xi_{\inf}(\gamma, \delta), \tag{5}$$

and

$$(\lambda\xi)_{\inf}(\gamma, \delta) = \lambda\xi_{\sup}(\gamma, \delta). \tag{6}$$

Proof: Without loss of generality, we only prove the first equation in each part.

If $\lambda = 0$, then the equation is obvious. When $\lambda > 0$, we have

$$\begin{aligned}
(\lambda\xi)_{\sup}(\gamma, \delta) &= \sup\{r \mid \mathrm{Ch}\{\lambda\xi \geq r\}(\gamma) \geq \delta\} \\
&= \lambda \sup\{r/\lambda \mid \mathrm{Ch}\{\xi \geq r/\lambda\}(\gamma) \geq \delta\} \\
&= \lambda\xi_{\sup}(\gamma, \delta).
\end{aligned}$$

A similar way may prove the other equation in part (a).

When $\lambda < 0$, we have

$$\begin{aligned}
(\lambda\xi)_{\sup}(\gamma, \delta) &= \sup\{r \mid \mathrm{Ch}\{-\lambda\xi \leq -r\}(\gamma) \geq \delta\} \\
&= -\inf\{(-r) \mid \mathrm{Ch}\{(-\lambda)\xi \leq (-r)\}(\gamma) \geq \delta\} \\
&= -(-\lambda)\xi_{\inf}(\gamma, \delta) \\
&= \lambda\xi_{\inf}(\gamma, \delta).
\end{aligned}$$

The rest equations in part (b) may be similarly proved. This completes the proof.

Theorem 3 *Let ξ be a birandom variable, and $\gamma, \delta \in (0, 1]$. Then*
(a) for fixed $\delta_0 \in (0, 1]$, $\xi_{\sup}(\gamma, \delta_0)$ is a decreasing and left-continuity function of γ;
(b) for fixed $\delta_0 \in (0, 1]$, $\xi_{\inf}(\gamma, \delta_0)$ is an increasing and left-continuity function of γ;
(c) for fixed $\gamma_0 \in (0, 1]$, $\xi_{\sup}(\gamma_0, \delta)$ is a decreasing and left-continuity function of δ;
(d) for fixed $\gamma_0 \in (0, 1]$, $\xi_{\inf}(\gamma_0, \delta)$ is an increasing and left-continuity function of δ.

Proof: (a) Assume that $\gamma_1, \gamma_2 \in (0, 1]$. If $\gamma_1 \leq \gamma_2$, then it follow immediately from the definitions of the (γ, δ_0)-pessimistic value that

$$\begin{aligned}
\xi_{\sup}(\gamma_1, \delta_0) &= \sup\{r | \mathrm{Ch}\{\xi \geq r\}(\gamma_1) \geq \delta_0\} \\
&\geq \sup\{r | \mathrm{Ch}\{\xi \geq r\}(\gamma_2) \geq \delta_0\} \\
&= \xi_{\sup}(\gamma_2, \delta_0).
\end{aligned}$$

Therefore, $\xi_{\sup}(\gamma, \delta_0)$ is a decreasing function of γ.

Next, we prove the left-continuity of $\xi_{\sup}(\gamma, \delta_0)$ with respect to γ. Let $\gamma_0 \in (0, 1]$ be given and $\{\gamma_n\}$ be an arbitrary sequence with $\gamma_n \uparrow \gamma_0$ as $n \to \infty$. It is sufficient to prove that $\lim_{n \to \infty} \xi_{\sup}(\gamma_n, \delta_0) = \xi_{\sup}(\gamma, \delta_0)$.

Write

$$\alpha_n = \xi_{\sup}(\gamma_n, \delta_0), \quad n = 0, 1, 2, \cdots$$

Since $\xi_{\sup}(\gamma, \delta_0)$ has been proved to be an decreasing function of γ, the sequence $\{\alpha_n\}$ is decreasing and $\alpha_n \geq \alpha_0$ holds for any positive integer n. Thus the limitation $\rho = \lim_{n \to \infty} \alpha_n$ exists and $\rho \geq \alpha_0$ holds.

On the other hand, note that the monotonicity of possibility measure and the continuity of ξ. By Theorem 1, we obtain that

$$\mathrm{Ch}\{\xi \geq \rho\}(\gamma_n) \geq \mathrm{Ch}\{\xi \geq \alpha_n\}(\gamma_n) \geq \delta_0.$$

Letting $n \to \infty$, by the left-continuity of chance function $\mathrm{Ch}\{\xi \geq x\}(\gamma)$ with respect to γ, we get $\mathrm{Ch}\{\xi \geq \rho\}(\gamma_0) \geq \delta_0$, which implies that $\rho \leq \alpha_0$. Hence $\rho = \alpha_0$ and thus $\xi_{\sup}(\gamma, \delta_0)$ is proved to be left-continuous.

(*b*) It follows from $\xi_{\inf}(\gamma, \delta_0) = -(-\xi)_{\sup}(\gamma, \delta_0)$ and the above proved (*a*) that $\xi_{\inf}(\gamma, \delta_0)$ is also a left-continuous function of γ.

(*c*) Assume that $\delta_1, \delta_2 \in (0, 1]$. If $\delta_1 \leq \delta_2$, then it follow immediately from the definitions of the (γ_0, δ)-optimistic value that

$$\xi_{\sup}(\gamma_0, \delta_1) = \sup\{r | \mathrm{Ch}\{\xi \geq r\}(\gamma_0) \geq \delta_1\}$$

$$\geq \sup\{r | \mathrm{Ch}\{\xi \geq r\}(\gamma_0) \geq \delta_2\}$$

$$= \xi_{\sup}(\gamma_0, \delta_2).$$

Therefore, $\xi_{\sup}(\gamma_0, \delta)$ is a decreasing function of δ.

Next, we prove the left-continuity of $\xi_{\sup}(\gamma_0, \delta)$ with respect to δ. Let $\delta_0 \in (0, 1]$ be given and $\{\delta_n\}$ be an arbitrary sequence with $\delta_n \uparrow \delta_0$ as $n \to \infty$. It is sufficient to prove that $\lim_{n \to \infty} \xi_{\sup}(\gamma_0, \delta_n) = \xi_{\sup}(\gamma_0, \delta_0)$.

Write

$$\beta_n = \xi_{\sup}(\gamma_0, \delta_n), \quad n = 0, 1, 2, \cdots$$

Since $\xi_{\sup}(\gamma_0, \delta)$ has been proved to be an decreasing function of δ, the sequence $\{\beta_n\}$ is decreasing and $\beta_n \geq \beta_0$ holds for any positive integer n. Thus the limitation $\rho = \lim_{n \to \infty} \beta_n$ exists and $\rho \geq \beta_0$ holds.

On the other hand, note that the monotonicity of possibility measure and the continuity of ξ. By Theorem 1, we obtain that

$$\mathrm{Ch}\{\xi \geq \rho\}(\gamma_0) \geq \mathrm{Ch}\{\xi \geq \beta_n\}(\gamma_0) \geq \delta_n.$$

Letting $n \to \infty$, we have $\mathrm{Ch}\{\xi \leq \rho\}(\gamma_0) \geq \delta_0$, which implies that $\rho \leq \beta_0$. Hence $\rho = \beta_0$ and thus $\xi_{\sup}(\gamma_0, \delta)$ is proved to be left-continuous.

(*d*) It follows from $\xi_{\inf}(\gamma_0, \delta) = -(-\xi)_{\sup}(\gamma_0, \delta)$ and the above proved (*c*) that $\xi_{\inf}(\gamma_0, \delta)$ is also a left-continuous function of δ.

Theorem 4 *Let ξ be a birandom variable, and $\gamma, \delta \in (0, 1]$. Then the following assertions are true:*
(*a*) *If $\gamma \leq 0.5$ and $\delta \leq 0.5$, then $\xi_{\inf}(\gamma, \delta) \leq \xi_{\sup}(\gamma, \delta)$;*
(*b*) *If $\gamma > 0.5$ and $\delta \geq 0.5$, then $\xi_{\inf}(\gamma, \delta) \geq \xi_{\sup}(\gamma, \delta)$.*

Proof: (*a*) Assume that $\gamma \leq 0.5$. For any given $\varepsilon > 0$, we define

$$\Omega_1 = \left\{\omega \in \Omega \mid \mathrm{Pr}\{\xi(\omega) > \xi_{\sup}(\gamma, \delta) + \varepsilon\} \geq \delta\right\},$$

$$\Omega_2 = \left\{\omega \in \Omega \mid \mathrm{Pr}\{\xi(\omega) < \xi_{\inf}(\gamma, \delta) - \varepsilon\} \geq \delta\right\}.$$

It follows from the definitions of $\xi_{\sup}(\gamma, \delta)$ and $\xi_{\inf}(\gamma, \delta)$ that $\Pr\{\Omega_1\} < \gamma$ and $\Pr\{\Omega_2\} < \gamma$. Thus $\Pr\{\Omega_1\} + \Pr\{\Omega_2\} < \gamma + \gamma \leq 1$. This fact implies that $\Omega_1 \cup \Omega_2 \neq \Omega$. Let $\omega^* \notin \Omega_1 \cup \Omega_2$. Then we have

$$\Pr\{\xi(\omega^*) > \xi_{\sup}(\gamma, \delta) + \varepsilon\} < \delta \text{ and } \Pr\{\xi(\omega^*) < \xi_{\inf}(\gamma, \delta) - \varepsilon\} < \delta.$$

Since $\Pr$ is self dual, we have

$$\Pr\{\xi(\omega^*) \leq \xi_{\sup}(\gamma, \delta) + \varepsilon\} > 1 - \delta \text{ and } \Pr\{\xi(\omega^*) \geq \xi_{\inf}(\gamma, \delta) - \varepsilon\} > 1 - \delta.$$

It follows from the definitions of $1-\delta$ optimistic value $\xi(\omega^*)_{\sup}(1-\delta)$ and $1-\delta$ pessimistic value $\xi(\omega^*)_{\inf}(1-\delta)$ of random variable $\xi(\omega^*)$ that

$$\xi_{\sup}(\gamma, \delta) + \varepsilon \geq \xi(\omega^*)_{\inf}(1 - \delta),$$

$$\xi_{\inf}(\gamma, \delta) - \varepsilon \leq \xi(\omega^*)_{\sup}(1 - \delta),$$

which implies that

$$\xi_{\inf}(\gamma, \delta) - \varepsilon - (\xi_{\sup}(\gamma, \delta) + \varepsilon)$$

$$\leq \xi(\omega^*)_{\sup}(1 - \delta) - \xi(\omega^*)_{\inf}(1 - \delta)$$

$$\leq 0$$

where the last inequality holds because $\delta \leq 0.5$. Letting $\varepsilon \to 0$, we obtain $\xi_{\inf}(\gamma, \delta) \leq \xi_{\sup}(\gamma, \delta)$.

(b) Assume $\gamma > 0.5$. For any given $\varepsilon > 0$, we define

$$\Omega_1 = \left\{\omega \in \Omega \mid \Pr\{\xi(\omega) \geq \xi_{\sup}(\gamma, \delta) - \varepsilon\} \geq \delta\right\},$$

$$\Omega_2 = \left\{\omega \in \Omega \mid \Pr\{\xi(\omega) \leq \xi_{\inf}(\gamma, \delta) + \varepsilon\} \geq \delta\right\}.$$

It follows from the definitions of $\xi_{\sup}(\gamma, \delta)$ and $\xi_{\inf}(\gamma, \delta)$ that $\Pr\{\Omega_1\} \geq \gamma$ and $\Pr\{\Omega_2\} \geq \gamma$. Thus $\Pr\{\Omega_1\} + \Pr\{\Omega_2\} \geq \gamma + \gamma > 1$. This fact implies that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Let $\omega^* \in \Omega_1 \cap \Omega_2$. Then we have

$$\Pr\{\xi(\omega^*) \geq \xi_{\sup}(\gamma, \delta) - \varepsilon\} \geq \delta,$$

$$\Pr\{\xi(\omega^*) \leq \xi_{\inf}(\gamma, \delta) + \varepsilon\} \geq \delta.$$

It follows from the definitions of $\xi(\omega^*)_{\sup}(\delta)$ and $\xi(\omega^*)_{\inf}(\delta)$ that

$$\xi_{\sup}(\gamma, \delta) - \varepsilon \leq \xi(\omega^*)_{\sup}(\delta),$$

$$\xi_{\inf}(\gamma, \delta) + \varepsilon \geq \xi(\omega^*)_{\inf}(\delta),$$

which implies that

$$\xi_{\sup}(\gamma, \delta) - \varepsilon - (\xi_{\inf}(\gamma, \delta) + \varepsilon) \leq \xi(\omega^*)_{\sup}(\delta) - \xi(\omega^*)_{\inf}(\delta) \leq 0$$

where the last inequality holds because $\delta \geq 0.5$. Letting $\varepsilon \to 0$, we obtain $\xi_{\inf}(\gamma, \delta) \geq \xi_{\sup}(\gamma, \delta)$.

Theorem 5 *Let ξ be a birandom variable, and $\{B_n\}$ a sequence of Borel sets of $\Re$ such that $B_n \downarrow B$. Then for $\alpha \in (0, 1]$, we have*

$$\lim_{n \to \infty} \text{Ch}\{\xi \in B_n\}(\alpha) = \text{Ch}\left\{\xi \in \lim_{n \to \infty} B_n\right\}(\alpha). \tag{7}$$

Proof: Write
$$\beta = \text{Ch}\{\xi \in B\}(\alpha), \quad \beta_n = \text{Ch}\{\xi \in B_n\}(\alpha), \quad n = 1, 2, \cdots$$

Since $B_n \downarrow B$, it is clear that $\beta_1 \geq \beta_2 \geq \cdots \geq \beta$. Thus the limitation
$$\rho = \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \text{Ch}\{\xi \in B_n\}(\alpha)$$

exists and $\rho \geq \beta$. On the other hand, since $\rho \leq \beta_n$ for each n and β_n is actually the optimistic value of the random variable $\Pr\{\xi(\omega) \in B_n\}$, it follows that

$$\Pr\{\omega \in \Omega \mid \Pr\{\xi(\omega) \in B_n\} \geq \rho\}$$
$$\geq \Pr\{\omega \in \Omega \mid \Pr\{\xi(\omega) \in B_n\} \geq \beta_n\}$$
$$\geq \alpha.$$

By using the probability continuity theorem, it is easy to verify that

$$\{\omega \in \Omega \mid \Pr\{\xi(\omega) \in B_n\} \geq \rho\} \downarrow \{\omega \in \Omega \mid \Pr\{\xi(\omega) \in B\} \geq \rho\}.$$

It follows again from the probability continuity theorem that

$$\Pr\{\omega \in \Omega \mid \Pr\{\xi(\omega) \in B\} \geq \rho\}$$
$$= \lim_{n \to \infty} \Pr\{\omega \in \Omega \mid \Pr\{\xi(\omega) \in B_n\} \geq \rho\}$$
$$\geq \alpha$$

which implies that $\rho \leq \beta$. Hence $\rho = \beta$ and (7) holds.

Theorem 6 *(a) Let $\xi, \xi_1, \xi_2, \cdots$ be birandom variables such that $\xi_n(\omega) \uparrow \xi(\omega)$ for each $\omega \in \Omega$. Then for any real number r and $\alpha \in (0, 1]$, we have*

$$\lim_{n \to \infty} \text{Ch}\{\xi_n \leq r\}(\alpha) = \text{Ch}\left\{\lim_{n \to \infty} \xi_n \leq r\right\}(\alpha). \tag{8}$$

(b) Let $\xi, \xi_1, \xi_2, \cdots$ be birandom variables such that $\xi_n(\omega) \downarrow \xi(\omega)$ for each $\omega \in \Omega$. Then for any real number r and $\alpha \in (0, 1]$, we have

$$\lim_{n \to \infty} \text{Ch}\{\xi_n \geq r\}(\alpha) = \text{Ch}\left\{\lim_{n \to \infty} \xi_n \geq r\right\}(\alpha). \tag{9}$$

Proof: (a) Write
$$\beta = \text{Ch}\{\xi \leq r\}(\alpha), \quad \beta_n = \text{Ch}\{\xi_n \leq r\}(\alpha), \quad n = 1, 2, \cdots$$

Since $\xi_n(\omega) \uparrow \xi(\omega)$ for each $\omega \in \Omega$, it is clear that $\{\xi_n(\omega) \leq r\} \downarrow \{\xi(\omega) \leq r\}$ for each $\omega \in \Omega$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta$. Thus the limitation
$$\rho = \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \text{Ch}\{\xi_n \leq r\}(\alpha)$$

exists and $\rho \geq \beta$. On the other hand, since $\rho \leq \beta_n$ for each n, we have

$$\Pr\{\omega \in \Omega \mid \Pr\{\xi_n(\omega) \leq r\} \geq \rho\}$$
$$\geq \Pr\{\omega \in \Omega \mid \Pr\{\xi_n(\omega) \leq r\} \geq \beta_n\}$$
$$\geq \alpha.$$

Since $\{\xi_n(\omega) \leq r\} \downarrow \{\xi(\omega) \leq r\}$ for each $\omega \in \Omega$, it follows from the probability continuity theorem that

$$\{\omega \in \Omega \mid \Pr\{\xi_n(\omega) \leq r\} \geq \rho\} \downarrow \{\omega \in \Omega \mid \Pr\{\xi(\omega) \leq r\} \geq \rho\}.$$

By using the probability continuity theorem again, we get

$$\Pr\{\omega \in \Omega \mid \Pr\{\xi(\omega) \leq r\} \geq \rho\}$$
$$= \lim_{n \to \infty} \Pr\{\omega \in \Omega \mid \Pr\{\xi_n(\omega) \leq r\} \geq \rho\}$$
$$\geq \alpha$$

which implies that $\rho \leq \beta$. Hence $\rho = \beta$ and
     The part (b) can be proved in a similar way.

Theorem 7 *Let $\gamma, \delta \in (0, 1]$ be given and $\xi, \xi_1, \xi_2, \cdots$ be birandom variables. We have the following results:*
(a) If $\xi_n \uparrow \xi$, then $\lim\limits_{n \to \infty} (\xi_n)_{\sup}(\gamma, \delta) \geq \xi_{\sup}(\gamma, \delta)$;
(b) If $\xi_n \downarrow \xi$, then $\lim\limits_{n \to \infty} (\xi_n)_{\inf}(\gamma, \delta) \leq \xi_{\inf}(\gamma, \delta)$.

Proof: (a) Since $\xi_n \uparrow \xi$, that is, $\xi_n(\omega) \uparrow \xi(\omega)$ for each $\omega \in \Omega$, it follows from Theorem 6 that for any $\gamma \in (0, 1]$, we have $\mathrm{Ch}\{\xi_n \leq r\}(\gamma) \downarrow \mathrm{Ch}\{\xi \leq r\}(\gamma)$. Hence, for any $\delta \in (0, 1]$, we have

$$\{r \mid \mathrm{Ch}\{\xi_n \leq r\}(\gamma) \geq \delta\} \downarrow \{r \mid \mathrm{Ch}\{\xi \leq r\}(\gamma) \geq \delta\}.$$

Therefore,

$$\lim_{n \to \infty} \sup \{r \mid \mathrm{Ch}\{\xi_n \leq r\}(\gamma) \geq \delta\} \geq \sup \{r \mid \mathrm{Ch}\{\xi \leq r\}(\gamma) \geq \delta\}.$$

That is, $\lim\limits_{n \to \infty} (\xi_n)_{\sup}(\gamma, \delta) \geq \xi_{\sup}(\gamma, \delta)$.
     (b) Since $\xi_n \downarrow \xi$, it follows from Theorem 6 that $\mathrm{Ch}\{\xi_n \geq r\}(\gamma) \downarrow \mathrm{Ch}\{\xi \geq r\}(\gamma)$ for any $\gamma \in (0, 1]$. Hence, for any $\delta \in (0, 1]$, we have

$$\{r \mid \mathrm{Ch}\{\xi_n \geq r\}(\gamma) \geq \delta\} \downarrow \{r \mid \mathrm{Ch}\{\xi \geq r\}(\gamma) \geq \delta\}.$$

Therefore,

$$\lim_{n \to \infty} \inf \{r \mid \mathrm{Ch}\{\xi_n \geq r\}(\gamma) \geq \delta\} \leq \inf \{r \mid \mathrm{Ch}\{\xi \geq r\}(\gamma) \geq \delta\}.$$

That is, $\lim\limits_{n \to \infty} (\xi_n)_{\inf}(\gamma, \delta) \leq \xi_{\inf}(\gamma, \delta)$. The theorem is proved.

4   Applications

There are several types of optimization models in birandom chance-constrained programming. With the help of above properties of optimistic and pessimistic values of birandom variable, we can discuss some relationships among the different models or analyze the bound or sensitivity of the solutions of birandom chance-constrained programming model.
     Assume that γ and δ, α_j and β_j are specified confidence levels for $j = 1, 2, \cdots, p$, x is the decision variable and ξ is the parametric birandom variable. Let $\max \overline{f}$ be the (γ, δ)-optimistic value to the return function $f(\boldsymbol{x}, \boldsymbol{\xi})$ and $\min \overline{f}$ be the (γ, δ)-pessimistic value to the return function $f(\boldsymbol{x}, \boldsymbol{\xi})$, respectively.

An analysis could be performed to see how the optimal objective value changes at various confidence levels. Given $\delta_1 \leq \delta_2$. Consider the following maximax birandom CCP models,

$$
\begin{cases}
\max \ \overline{f} \\
\text{subject to:} \\
\quad \text{Ch}\left\{ f(\boldsymbol{x}, \boldsymbol{\xi}) \geq \overline{f} \right\}(\gamma) \geq \delta_1 \\
\quad \text{Ch}\left\{ g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0 \right\}(\alpha_j) \geq \beta_j \\
\quad j = 1, 2, \cdots, p,
\end{cases}
\tag{10}
$$

$$
\begin{cases}
\max \ \overline{f} \\
\text{subject to:} \\
\quad \text{Ch}\left\{ f(\boldsymbol{x}, \boldsymbol{\xi}) \geq \overline{f} \right\}(\gamma) \geq \delta_2 \\
\quad \text{Ch}\left\{ g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0 \right\}(\alpha_j) \geq \beta_j \\
\quad j = 1, 2, \cdots, p.
\end{cases}
\tag{11}
$$

From Theorem 3, we have

Theorem 8 *If $\delta_1 \leq \delta_2$, then the optimal objective value of* (10) *is larger than or equal to that of* (11).

Take another example. Consider the following maximax birandom CCP model,

$$
\begin{cases}
\max \max \ \overline{f} \\
\text{subject to:} \\
\quad \text{Ch}\left\{ f(\boldsymbol{x}, \boldsymbol{\xi}) \geq \overline{f} \right\}(\gamma) \geq \delta \\
\quad \text{Ch}\left\{ g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0 \right\}(\alpha_j) \geq \beta_j \\
\quad j = 1, 2, \cdots, p
\end{cases}
\tag{12}
$$

and the following minimax birandom CCP model,

$$
\begin{cases}
\max \min \ \overline{f} \\
\text{subject to:} \\
\quad \text{Ch}\left\{ f(\boldsymbol{x}, \boldsymbol{\xi}) \leq \overline{f} \right\}(\gamma) \geq \delta \\
\quad \text{Ch}\left\{ g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0 \right\}(\alpha_j) \geq \beta_j \\
\quad j = 1, 2, \cdots, p.
\end{cases}
\tag{13}
$$

From Theorem 4, we have

Theorem 9 *If $\gamma > 0.5$ and $\delta \geq 0.5$, then the optimal objective value of* (12) *is smaller than or equal to that of* (13).

5   Conclusions

In this paper we study two types of the critical values of birandom variables. Some properties of optimistic and pessimistic values of birandom variable are investigated. The discussed results are expected to use in parameter analysis of birandom programming.

Acknowledgments

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