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Uniform Convergence of Fuzzy Random Sequence

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Abstract. Fuzzy random variable is defined as a measurable function from the probability space to the set of fuzzy numbers. In this paper, the concept of uniform convergence of fuzzy random sequence is presented. Additionally, some mathematical properties of convergent fuzzy random sequence are also obtained.

Keywords: Fuzzy Number; Fuzzy Random Variable; Uniform Convergence

1 Introduction

The concept of fuzzy random variable was introduced as a mathematical description for fuzzy stochastic phenomenon. Kwakernaak[6][7] first introduced this notion. And then, fuzzy random theory was further developed by several researchers such as Kruse and Meyer[5], Puri and Ralescu [8], Liu and Liu[9], and so on.

In Puri and Ralescu's(ref. [8]) definition, a fuzzy random variable is a measurable fuzzy mapping from a probability space $(\Omega, A, \Pr)$ to the set of fuzzy numbers, where the measurability condition refers to so called "graph-levelwise measurability". As for the measurability condition of fuzzy mapping, many researchers presented several different definitions such as "measurability", "levelwise measurability", "strong measurability", "weak measurability", and so on. For more details, interested readers may refer to Rockefellar[10], Kaleva[3], and so on. Additionally, Butnariu[1] and Wu et. al[11] studied different measurability concepts and shown that some of these concepts are equivalent or strongly inter-related. That is, if the probability space is completed, "graph-levelwise measurability", "measurability", "levelwise measurability" and "strong measurability" are equivalent. Thus, the definition of fuzzy random variable can be expressed in several ways if the probability space is completed.

The purpose of this paper is to study uniform convergence of fuzzy random sequence. In Section 2, we first discuss some mathematical properties of fuzzy numbers sequence. Then in Section 3, uniform convergence of fuzzy random sequence is defined and some relevant mathematical properties are also obtained.

2 Preliminaries

A fuzzy number is a fuzzy set $\xi : \Re \to [0, 1]$ with the following properties:
(i) ξ is normal, that is, there exists $x_0 \in \Re$ such that the membership function $\mu_\xi(x_0) = 1$;
(ii) ξ is upper semicontinuous;
(iii) ξ is a convex fuzzy set, that is, $\mu_\xi(\lambda x + (1 - \lambda)y) \geq \min\{\mu_\xi(x), \mu_\xi(y)\}$ for $x, y \in \Re$ and $\lambda \in [0, 1]$;

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(iv) supp ξ= cl$\{x \in \Re | \mu_\xi(x) > 0\}$ is compact.

In this paper, the set of fuzzy numbers is denoted by $F(\Re)$. For a fuzzy number ξ, the λ-cut set is defined as

$$\xi_\lambda = \begin{cases} \{x : \mu_\xi(x) \geq \lambda\}, & \text{if } 0 < \lambda \leq 1 \\ \text{supp } \xi, & \text{if } \lambda = 0. \end{cases}$$

It is true that ξ is a fuzzy number if and only if ξ_λ is a closed bounded interval for each $\lambda \in [0, 1]$ and $\xi_1 \neq \phi$. In the following, we shall use the notation $\xi_\lambda = [\xi_\lambda^-, \xi_\lambda^+]$ to denote the λ-cut set of fuzzy number ξ.

Theorem 2.1 *[2] Let $\xi \in F(\Re)$. Denote $\xi^-(\lambda) = \xi_\lambda^-$, $\xi^+(\lambda) = \xi_\lambda^+$. Then*
(i) $\xi^-(\lambda)$ is a bounded increasing function on $[0, 1]$;
(ii) $\xi^+(\lambda)$ is a bounded decreasing function on $[0, 1]$;
(iii) $\xi^-(1) \leq \xi^+(1)$;
(iv) $\xi^-(\lambda)$ and $\xi^+(\lambda)$ are left-continuous on $(0, 1]$ and right-continuous at 0;
(v) If $\xi^-(\lambda)$ and $\xi^+(\lambda)$ satisfy above (i)-(iv), then there exists a unique $\eta \in F(\Re)$ such that $\eta_\lambda^- = \xi^-(\lambda)$ and $\eta_\lambda^+ = \xi^+(\lambda)$.

Definition 2.1 *Let ξ and η be two fuzzy numbers. We say that ξ is larger than or equal to η if for any $\lambda \in [0, 1]$, we have $\xi_\lambda^- \geq \eta_\lambda^-$ and $\xi_\lambda^+ \geq \eta_\lambda^+$, denoted by "$\xi \geq \eta$" or "$\eta \leq \xi$".*

In order to measure the distance between two fuzzy numbers, we use the following metric d: let $\xi, \eta \in F(\Re)$, then

$$d(\xi, \eta) = \sup_{0 \leq \lambda \leq 1} d_H(\xi_\lambda, \eta_\lambda),$$

where d_H is the Hausdorff metric defined as

$$d_H(\xi_\lambda, \eta_\lambda) = \max\{|\xi_\lambda^- - \eta_\lambda^-|, |\xi_\lambda^+ - \eta_\lambda^+|\}.$$

Also, $d(\xi, 0)$ will be denoted by $\|\xi\|$.

Definition 2.2 *Let $\xi, \xi_1, \xi_2, \cdots, \xi_n, \cdots$ be a sequence of fuzzy numbers. We call that the sequence $\{\xi_i\}$ converges to ξ if*

$$\lim_{i \to \infty} d(\xi_i, \xi) = 0.$$

It is clear that a sequence $\{\xi_i\}$ of fuzzy numbers converges to ξ if and only if $\lim\limits_{i \to \infty} (\xi_i)_\lambda^- = \xi_\lambda^-$ and $\lim\limits_{i \to \infty} (\xi_i)_\lambda^+ = \xi_\lambda^+$ uniformly with respect to $\lambda \in [0, 1]$.

Theorem 2.2 *Let $\{\xi_i\}$ be a sequence of fuzzy numbers. If $\lim\limits_{i \to \infty} (\xi_i)_\lambda^- = \xi_\lambda^-$ and $\lim\limits_{i \to \infty} (\xi_i)_\lambda^+ = \xi_\lambda^+$ uniformly with respect to $\lambda \in [0, 1]$, then $\{\xi_i\}$ converges to a fuzzy number ξ, whose λ-cut set is $\xi_\lambda = [\xi_\lambda^-, \xi_\lambda^+]$ for any $\lambda \in [0, 1]$.*

Proof. Denote $\xi^-(\lambda) = \xi_\lambda^-, \xi^+(\lambda) = \xi_\lambda^+$. It is easy to prove that $\xi^-(\lambda)$ is a bounded increasing function on $[0, 1]$ and $\xi^+(\lambda)$ is a bounded decreasing function on $[0, 1]$. Note that $(\xi_i)^-(1) \leq (\xi_i)^+(1)$ for each i. Taking limits on both sides, we have $\xi^-(1) \leq \xi^+(1)$.

Next, we prove that $\xi^-(\lambda)$ and $\xi^+(\lambda)$ are left-continuous on $(0, 1]$. Let $\lambda_0 \in (0, 1]$ be given. Since $\lim\limits_{i \to \infty} (\xi_i)_\lambda^- = \xi_\lambda^-$ uniformly with respect to $\lambda \in [0, 1]$, it follows that for any given $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$|(\xi_i)^-(\lambda) - \xi^-(\lambda)| < \frac{\varepsilon}{3}$$

for all $i \geq N$ and $\lambda \in [0,1]$. Then for a given $i_0 \geq N$, we have

$$|(\xi_{i_0})^-(\lambda) - \xi^-(\lambda)| < \frac{\varepsilon}{3}, \ \ |(\xi_{i_0})^-(\lambda_0) - \xi^-(\lambda_0)| < \frac{\varepsilon}{3}.$$

On the other hand, since $(\xi_i)^-(\lambda)$ is a left-continuous function of $\lambda \in (0,1]$, for the above ε, there exists $\delta > 0$ such that

$$|(\xi_{i_0})^-(\lambda) - (\xi_{i_0})^-(\lambda_0)| < \frac{\varepsilon}{3}$$

for all λ with $0 \leq \lambda_0 - \lambda < \delta$. Then we have

$$\begin{aligned} |\xi^-(\lambda) - \xi^-(\lambda_0)| \ &\leq \ |(\xi_{i_0})^-(\lambda) - \xi^-(\lambda)| + |(\xi_{i_0})^-(\lambda) - (\xi_{i_0})^-(\lambda_0)| \\ &\quad + |(\xi_{i_0})^-(\lambda_0) - \xi^-(\lambda_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for all λ with $0 \leq \lambda_0 - \lambda < \delta$, which proves the left-continuity of $\xi^-(\lambda)$. The left-continuity of $\xi^+(\lambda)$ can be proven similarly.

A similar way may prove the right-continuity at 0 of $\xi^-(\lambda)$ and $\xi^+(\lambda)$. Thus, it follows from Theorem 2.1 that there exists a unique fuzzy number $\xi \in F(\Re)$ such that the λ-cut set is $\xi_\lambda = [\xi_\lambda^-, \xi_\lambda^+]$ for any $\lambda \in [0,1]$, which implies that $\{\xi_i\}$ converges to ξ. The proof is completed.

Theorem 2.2 states that the fuzzy numbers sequence $\{\xi_i\}$ is convergent if and only if $(\xi_i)_\lambda^-$ and $(\xi_i)_\lambda^+$ converge uniformly with respect to $\lambda \in [0,1]$ as $i \to \infty$.

3 Uniform Convergence of Fuzzy Random Sequence

Throughout this paper, $(\Omega, \boldsymbol{A}, \Pr)$ denotes a completed probability space. The function $\xi : \Omega \to F(\Re)$ is called fuzzy random variable if for every closed subset B of $\Re$, the fuzzy set $\xi^{-1}(B)$ is measurable when considered as a function from Ω to $[0,1]$, where $\xi^{-1}(B)$ denotes the fuzzy subset of Ω defined by

$$\xi^{-1}(B)(\omega) = \sup_{x \in B} \xi(\omega)(x)$$

for every $\omega \in \Omega$. Kim and Ghil[4] also proved that ξ is a fuzzy random variable if and only if for each $\lambda \in [0,1]$, $\xi_\lambda^-(\omega)$ and $\xi_\lambda^+(\omega)$ are random variables when considered as the functions from Ω to $\Re$, where $\xi_\lambda^-(\omega)$ and $\xi_\lambda^+(\omega)$ are end points of the λ-cut set of fuzzy number $\xi(\omega)$.

Definition 3.1 *Let $\xi, \xi_1, \xi_2, \cdots$ be a sequence of fuzzy random variables. We say that $\{\xi_i\}$ converges to ξ uniformly if there exists a set $A \in \boldsymbol{A}$ with $\Pr\{A\} = 1$ such that*

$$\lim_{i\to\infty} \sup_{\omega \in A} d(\xi_i(\omega), \xi(\omega)) = 0.$$

Theorem 3.1 *Let $\xi, \xi_1, \xi_2, \cdots$ be a sequence of fuzzy random variables. Then $\{\xi_i\}$ converges to ξ uniformly if and only if there exists a set $A \in \boldsymbol{A}$ with $\Pr\{A\} = 1$ such that $\lim_{i\to\infty} (\xi_i)_\lambda^-(\omega) = \xi_\lambda^-(\omega)$ and $\lim_{i\to\infty} (\xi_i)_\lambda^+(\omega) = \xi_\lambda^+(\omega)$ uniformly with respect to $\omega \in A$ and $\lambda \in [0,1]$.*

Proof. Since $\{\xi_i\}$ converges to ξ uniformly, it follows from Definition 3.1 that there exists a set $A \in \boldsymbol{A}$ with $\Pr\{A\} = 1$ such that

$$\lim_{i\to\infty} \sup_{\omega \in A} d(\xi_i(\omega), \xi(\omega)) = 0.$$

Thus, we have

$$\lim_{i\to\infty} \sup_{\omega \in A} \sup_{0\leq\lambda\leq 1} \max\{|(\xi_i)_\lambda^-(\omega) - \xi_\lambda^-(\omega)|, \ |(\xi_i)_\lambda^+(\omega) - \xi_\lambda^+(\omega)|\} = 0,$$

which implies that

$$\lim_{i\to\infty}\sup_{\omega\in A}\sup_{0\le\lambda\le1}|(\xi_i)^-_\lambda(\omega)-\xi^-_\lambda(\omega)|=0,$$

$$\lim_{i\to\infty}\sup_{\omega\in A}\sup_{0\le\lambda\le1}|(\xi_i)^+_\lambda(\omega)-\xi^+_\lambda(\omega)|=0.$$

Thus, $\lim_{i\to\infty}(\xi_i)^-_\lambda(\omega)=\xi^-_\lambda(\omega)$ and $\lim_{i\to\infty}(\xi_i)^+_\lambda(\omega)=\xi^+_\lambda(\omega)$ uniformly with respect to $\omega\in A$ and $\lambda\in[0,1]$.

Conversely, if there exists a set $A\in\boldsymbol{A}$ with $\Pr\{A\}=1$ such that $\lim_{i\to\infty}(\xi_i)^-_\lambda(\omega)=\xi^-_\lambda(\omega)$ and $\lim_{i\to\infty}(\xi_i)^+_\lambda(\omega)=\xi^+_\lambda(\omega)$ uniformly with respect to $\omega\in A$ and $\lambda\in[0,1]$, then for any real number $\varepsilon>0$, there exists an integer $N>0$ such that

$$|(\xi_i)^-_\lambda(\omega)-\xi^-_\lambda(\omega)|<\varepsilon,\quad |(\xi_i)^+_\lambda(\omega)-\xi^+_\lambda(\omega)|<\varepsilon$$

for all $i\ge N$, $\omega\in A$ and $\lambda\in[0,1]$. Therefore, we have

$$\sup_{\omega\in A}\sup_{0\le\lambda\le1}|(\xi_i)^-_\lambda(\omega)-\xi^-_\lambda(\omega)|\le\varepsilon,\quad \sup_{\omega\in A}\sup_{0\le\lambda\le1}|(\xi_i)^+_\lambda(\omega)-\xi^+_\lambda(\omega)|\le\varepsilon.$$

It follows that

$$\sup_{\omega\in A}\sup_{0\le\lambda\le1}\max\{|(\xi_i)^-_\lambda(\omega)-\xi^-_\lambda(\omega)|,\ |(\xi_i)^+_\lambda(\omega)-\xi^+_\lambda(\omega)|\}\le\varepsilon$$

for all $i\ge N$. That is,

$$\lim_{i\to\infty}\sup_{\omega\in A}d(\xi_i(\omega),\xi(\omega))=0.$$

The proof is thus completed.

Theorem 3.2 *Let $\xi_1,\xi_2,\cdots$ be a sequence of fuzzy random variables. If there exists a set $A\in\boldsymbol{A}$ with $\Pr\{A\}=1$ such that $(\xi_i)^-_\lambda(\omega)$ and $(\xi_i)^+_\lambda(\omega)$ converge uniformly with respect to $\omega\in A$ and $\lambda\in[0,1]$ as $i\to\infty$, then $\{\xi_i\}$ converges uniformly to fuzzy random variable ξ whose λ-cut set is $\xi_\lambda(\omega)=\left[\lim_{i\to\infty}(\xi_i)^-_\lambda(\omega),\lim_{i\to\infty}(\xi_i)^+_\lambda(\omega)\right]$ for each $\omega\in A$.*

Proof. It follows from Theorem 2.2 that for each $\omega\in A$, we can obtain a fuzzy number $\bar{\xi}(\omega)$ with the λ-cut set $[\lim_{i\to\infty}(\xi_i)^-_\lambda(\omega),\ \lim_{i\to\infty}(\xi_i)^+_\lambda(\omega)]$. Now, we construct a fuzzy number valued function ξ:

$$\xi(\omega)=\begin{cases}\bar{\xi}, & \text{if } \omega\in A\\ 0, & \text{otherwise.}\end{cases}$$

It is clear that

$$\xi^-_\lambda(\omega)=\begin{cases}\lim_{i\to\infty}(\xi_i)^-_\lambda(\omega), & \text{if } \omega\in A\\ 0, & \text{otherwise,}\end{cases}$$

$$\xi^+_\lambda(\omega)=\begin{cases}\lim_{i\to\infty}(\xi_i)^+_\lambda(\omega), & \text{if } \omega\in A\\ 0, & \text{otherwise.}\end{cases}$$

Since $(\xi_i)^-_\lambda(\omega)$ and $(\xi_i)^+_\lambda(\omega)$ are random variables when considered as the functions from Ω to $\Re$, we can easily prove that $\xi^-_\lambda(\omega)$ and $\xi^+_\lambda(\omega)$ are random variables, which implies that ξ is a fuzzy random variable. By using Theorem 3.1, we obtain that the sequence $\{\xi_i\}$ converges to ξ uniformly. The proof is thus completed.

Theorem 3.3 *Let $\xi,\xi_1,\xi_2,\cdots$ be a sequence of fuzzy random variables. If $\{\xi_i\}$ converges uniformly to ξ, then there exists a set $A\in\boldsymbol{A}$ with $\Pr\{A\}=1$ such that $\lim_{i\to\infty}\int_0^1(\xi_i)^-_\lambda(\omega)\mathrm{d}\lambda=\int_0^1\xi^-_\lambda(\omega)\mathrm{d}\lambda$ and $\lim_{i\to\infty}\int_0^1(\xi_i)^+_\lambda(\omega)\mathrm{d}\lambda=\int_0^1\xi^+_\lambda(\omega)\mathrm{d}\lambda$ uniformly with respect to $\omega\in A$.*

Proof. It follows from uniform convergence of $\{\xi_i\}$ that there exists a set $A \in \boldsymbol{A}$ with $\Pr\{A\} = 1$ such that $(\xi_i)^-_\lambda(\omega)$ and $(\xi_i)^+_\lambda(\omega)$ converge to $\xi^-_\lambda(\omega)$ and $\xi^+_\lambda(\omega)$ uniformly with respect to $\omega \in A$ and $\lambda \in [0,1]$ as $i \to \infty$, respectively. Then for any $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$|(\xi_i)^-_\lambda(\omega) - \xi^-_\lambda(\omega)| < \varepsilon, \quad |(\xi_i)^+_\lambda(\omega) - \xi^+_\lambda(\omega)| < \varepsilon$$

for any $i \geq N$, $\omega \in A$ and $\lambda \in [0,1]$. Hence, we have

$$\left|\int_0^1 (\xi_i)^-_\lambda(\omega)\mathrm{d}\lambda - \int_0^1 \xi^-_\lambda(\omega)\mathrm{d}\lambda\right| \leq \int_0^1 \left|(\xi_i)^-_\lambda(\omega) - \xi^-_\lambda(\omega)\right| \mathrm{d}\lambda < \varepsilon,$$

$$\left|\int_0^1 (\xi_i)^+_\lambda(\omega)\mathrm{d}\lambda - \int_0^1 \xi^+_\lambda(\omega)\mathrm{d}\lambda\right| \leq \int_0^1 \left|(\xi_i)^+_\lambda(\omega) - \xi^+_\lambda(\omega)\right| \mathrm{d}\lambda < \varepsilon$$

for any $i \geq N$ and $\omega \in A$, which implies that $\lim_{i\to\infty} \int_0^1 (\xi_i)^-_\lambda(\omega)\mathrm{d}\lambda = \int_0^1 \xi^-_\lambda(\omega)\mathrm{d}\lambda$ and $\lim_{i\to\infty} \int_0^1 (\xi_i)^+_\lambda(\omega)\mathrm{d}\lambda = \int_0^1 \xi^+_\lambda(\omega)\mathrm{d}\lambda$ uniformly with respect to $\omega \in A$. The proof is completed.

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