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Approach Algorithm Research and Theorem Proof Based on Quadratic B-Spline Curves

Jiefu Wang, Qiang Li, Yong Jiang +, Jianliang Li

Department of Mathematics, Nanjing University of Science and Technology Nanjing, 210094, P. R. China

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Abstract. Reference [1] presents an efficient and sufficient algorithm to approximate general interpolating Spline curve which goes through the given set of design-data points by using asymptotic iterative B-spline curves. Firstly, the given set is considered as the control points of a B-spline, to create the initial approximate curve, then the iterative grade is built based on the error between the initial approximate curve and given set, it is used to generate an iterative function sequence to approximate the interpolating function of the cover. Two theorems were given in reference [1], but didn't get proved. In this text, we give a general proof based on the preceding context and several other references. In the meantime, numerical experiments further prove that this algorithm is really simple and valid.

Key words: Spline function, iterative algorithm, theorem proof, numerical experiment

1. Introduction

The spline function has many extensive applications, reference [1] put forward a algorithm which used iterative B spline curve to produce design-data points. This algorithm can be successfully applied in 2-D, 3-D sketches, the computer graphical, complex model in CAD and CAM. The text gave the super-vector sequence's astringency theorem (theorem 3) and the error sequence's astringency theorem (theorem 2),but did not give their proofs.

This text gave a general proof of the two theorems on the foundation of the algorithm. Firstly, prove the theorem under the condition that the control points are equidistant. The thought is get its iterative matrix A by the algorithm, and complete theorem 2 and 3 proof using $\rho(I-A)<1$; Secondly, prove the theorem under the general condition, namely the control points are equidistant or unequidistant. Its thought and process are similar to the first item, and it can be regarded as the first item's general extension; Finally, give the numerical experiment of the algorithm. And further verify that the algorithm is easy and efficient. In order to make the proof simple, we just give the proof on the equidistant condition.

2. algorithm introduction

2.1 problem description

Now, we introduce the algorithm in reference [1] briefly. (note: The algorithm provided by reference [1] is on the equidistant condition).The concrete content can be got in reference[1].

Suppose super vector h

$$h=(\overrightarrow{h_1}\ \overrightarrow{h_2}\cdots\ \overrightarrow{h_i}\cdots\overrightarrow{h_N})=\begin{pmatrix} h_1^x & h_2^x & \cdots & h_i^x & \cdots & h_N^x \\ h_1^x & h_2^y & \cdots & h_i^y & \cdots & h_N^y \\ h_1^x & h_2^z & \cdots & h_i^z & \cdots & h_N^z \end{pmatrix}$$

is a set of design-data points in the 3D space。We hope to find a curve $s(t)$ that pass all the points above in 3D space, and we request $s(t)$ has better smooth.

+ Corresponding author. *Email*: math@mail.njust.edu.cn

2.2 Add boundary data

If the data points are periodic, we can get super vector

$$h^* = (\overrightarrow{h_0} | \overrightarrow{h_1}\ \overrightarrow{h_2} \cdots \overrightarrow{h_i} \cdots \overrightarrow{h_N} | \overrightarrow{h_{N+1}})$$

by adding boundary data $\vec{h}_0$ 、 $\vec{h}_{N+1}$ .Where $\vec{h}_{N+1} = \vec{h}_1, \vec{h}_0 = \vec{h}_N$

2.3 algorithm explanation

Iterative process of the iterative algorithm provided in reference [1] can be seen in the following figure(The dotted line represents the $(k\text{-}1)_{th}$ iterative line and the solid line represents the $(k)_{th}$ iterative line).

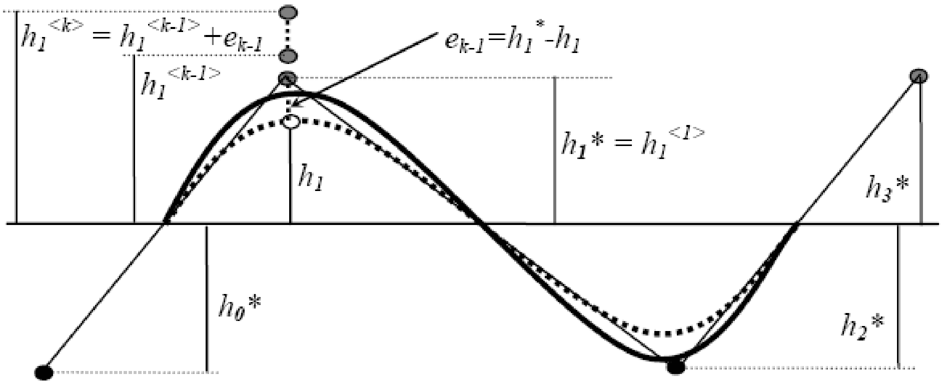

Fig.1 iterative process sketch map

Concrete algorithm is as follows:

Step 1: Using h* as the initial set of control points marked as $h^{<1>}(k=1)$. Find out the B-spline $\vec{s}^{<1>}(t)$ by using the parabola B-spline which is described by formula as following:

$$\vec{s}^{<1>}(t) = \vec{s}_i^{<1>}(t) = \frac{1}{2}\begin{pmatrix}\vec{h}_{i-1}^{<1>} & \vec{h}_i^{<1>} & \vec{h}_{i+1}^{<1>}\end{pmatrix}\begin{pmatrix}1 & -2 & 1 \\ 1 & 2 & -2 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 \\ t \\ t^2\end{pmatrix}$$

$(0 \le t \le 1, i = 1,2,\cdots N)$,

Where

$$\vec{s}_i(t) = \begin{pmatrix}x_i(t) \\ y_i(t) \\ z_i(t)\end{pmatrix} \quad \begin{pmatrix}\vec{h}_{i-1}^{<k>} & \vec{h}_i^{<k>} & \vec{h}_{i+1}^{<k>}\end{pmatrix} = \begin{pmatrix}h_{i-1}^{x,<k>} & h_i^{x,<k>} & h_{i+1}^{x,<k>} \\ h_{i-1}^{y,<k>} & h_i^{y,<k>} & h_{i+1}^{y,<k>} \\ h_{i-1}^{z,<k>} & h_i^{z,<k>} & h_{i+1}^{z,<k>}\end{pmatrix}.$$

Step 2: use

$$h^{*<1>} = \left(h_0^{*<1>} \quad | \quad h_1^{*<1>} \quad h_2^{*<1>} \quad \cdots \quad h_i^{*<1>} \quad \cdots \quad h_N^{*<1>} \quad | \quad h_{N+1}^{*<1>}\right)$$

to record the super-vector of values of B-spline $\vec{s}^{\langle 1 \rangle}(t)$ which respect to control points $h^{<1>}$. For equidistant nodes,

where

$$h_i^{*<1>} = \vec{s}_i^{<1>}(\frac{1}{2}) \quad (i = 1,2,\cdots N),$$

And

$$h_0^{*<k>} = h_N^{*<k>}, \quad h_{N+1}^{*<k>} = h_1^{*<k>}$$

Step 3: Calculate the error super vector $e^{<1>} = h^* - h^1$.And judge whether the error achieves the accuracy request or not. If achieves, stop the procedure; If not, turn to the fourth step.

Step 4: Calculate new control points: $h^{<2>} = h^{<1>} + e$, and circulate step 1, 2 and 3

Note: We record the control points that the $(k)_{th}$ circulation needed as $h^{<k>}$ and the curve we get as $\vec{s}^{\langle k\rangle}(t)$. Record the super-vector of values of B-spline $\vec{s}^{\langle k\rangle}(t)$ with respect to control points $h^{<k>}$ as $h*^{<k>}$, and record error vector as $e^{\langle k\rangle}$.

3. Proofs of theorem 2 and 3

3.1. theorem 2.

Error sequence $e^{\langle k\rangle}$ is a "gradually decrease super vector sequence" ,Namely

$$\left\|e^{\langle 1\rangle}\right\| \geq \left\|e^{\langle 2\rangle}\right\| \geq \cdots \left\|e^{\langle k\rangle}\right\| \geq \cdots$$

3.2. theorem 3.

Super vector sequence $h^{*\langle k\rangle}(k = 1,2,\cdots)$ is converged at h^*, Namely

$$\lim_{k\to\infty}\left\|e^{\langle k\rangle}\right\| = \lim_{k\to\infty}\left\|h^* - h^{*\langle k\rangle}\right\| = \lim_{k\to\infty}\left(h^* - h^{*\langle k\rangle}\right)^T\left(h^* - h^{*\langle k\rangle}\right) = 0$$

And

$$\lim_{k\to\infty} h^{*\langle k\rangle} = h^*$$

The theorems indicated that $h^{*<k>}$ and $e^{<k>}$ converge at h* and 0 respectively. That means the B-spline sequence $\vec{s}^{\langle k\rangle}(t)$ is approximate to special piecewise curve $\vec{s}(t)$ which go through the design-data points. The curve can be called as “interpolating spline”.

Note: Here, equality($h_0^{\langle k\rangle} = h_N^{\langle k\rangle}$, $h_{N+1}^{\langle k\rangle} = h_1^{\langle k\rangle}$) means that we don't consider the meaning of the first sits x. For example

$$\begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} \text{and} \begin{pmatrix} 6 \\ -4 \\ 0 \end{pmatrix}$$

are equal.

4.

4.1When the control points are equidistant

4.1.1 Proof of theorem 3

We can know from the algorithm:

$$h_i^{*<k>} = s_i^{<k>}(\frac{1}{2})$$
$$= \frac{1}{2}(h_{i-1}^{<k>} \quad h_i^{<k>} \quad h_{i+1}^{<k>})\begin{pmatrix} 1 & -2 & 1 \\ 1 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$$
$$= \frac{1}{8}h_{i-1}^{<k>} + \frac{3}{4}h_i^{<k>} + \frac{1}{8}h_{i+1}^{<k>}$$

$i = 1,2,\cdots,N$, k means the iterative time.

because $h_0^{*<k>} = h_N^{*<k>}$, $h_{N+1}^{*<k>} = h_1^{*<k>}$ then

$$h_0^{*<k>}$$
$$= h_N^{*<k>} = \frac{1}{8}h_{N-1}^{<k>} + \frac{3}{4}h_N^{<k>} + \frac{1}{8}h_{N+1}^{<k>}$$
$$= \frac{1}{8}h_{N-1}^{<k>} + \frac{3}{4}h_0^{<k>} + \frac{1}{8}h_1^{<k>}$$
$$= \frac{3}{4}h_0^{<k>} + \frac{1}{8}h_1^{<k>} + \frac{1}{8}h_{N-1}^{<k>}$$
$$h_{N+1}^{*<k>}$$
$$= h_1^{*<k>} = \frac{1}{8}h_0^{<k>} + \frac{3}{4}h_1^{<k>} + \frac{1}{8}h_2^{<k>}$$
$$= \frac{1}{8}h_N^{<k>} + \frac{3}{4}h_{N+1}^{<k>} + \frac{1}{8}h_2^{<k>}$$

Then, iterative matrix A and control points sequence $h^{\langle k \rangle}$ can be denoted as

$$A = \begin{bmatrix} \frac{3}{4} & \frac{1}{8} & 0 & 0 & \cdots & \frac{1}{8} & 0 & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{8} & 0 & \cdots & 0 & \frac{1}{8} & \frac{3}{4} \end{bmatrix}$$

$$h^{\langle k \rangle} = \left(h_N^{\langle k \rangle}, h_1^{\langle k \rangle}, h_2^{\langle k \rangle}, \cdots, h_{N-1}^{\langle k \rangle}, h_N^{\langle k \rangle}, h_1^{\langle k \rangle}\right)^T \quad (k = 1,2,\cdots)$$

Where matrix A is N+2 rank strict principal diagonal excellent matrix, so A is inversible.

According to the algorithm, we know

$$h^{*\langle k \rangle} = Ah^{\langle k \rangle}$$

Then

$$\begin{aligned}
h^{\langle k\rangle} &= h^{\langle k-1\rangle} + e^{\langle k-1\rangle} \\
&= h^{\langle k-1\rangle} + (h^{*} - h^{*\langle k-1\rangle}) \\
&= h^{\langle k-1\rangle} + (h^{*} - Ah^{\langle k-1\rangle}) \\
&= (I-A)h^{\langle k-1\rangle} + h^{*} \\
&= (I-A)[(I-A)h^{\langle k-2\rangle} + h^{*}] + h^{*} \\
&= (I-A)^{2} h^{\langle k-2\rangle} + (I-A)h^{*} + h^{*} \\
&= \cdots\cdots \\
&= (I-A)^{k-1} h^{\langle 1\rangle} + (I-A)^{\langle k-2\rangle} h^{*} + \cdots + (I-A)h^{*} + h^{*} \\
&= (I-A)^{k-1} h^{*} + (I-A)^{\langle k-2\rangle} h^{*} + \cdots + (I-A)h^{*} + h^{*} \\
&= \left[(I-A)^{k-1} + (I-A)^{k-2} + \cdots + (I-A) + I\right] h^{*} \\
&= \left[I - (I-A)^{k}\right][I-(I-A)]^{-1} h^{*} \\
&= \left[I - (I-A)^{k}\right] A^{-1} h^{*}
\end{aligned}$$

And because

$$I - A = \begin{bmatrix} \frac{1}{4} & -\frac{1}{8} & 0 & 0 & \cdots & -\frac{1}{8} & 0 & 0 \\ -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} \\ 0 & 0 & -\frac{1}{8} & 0 & \cdots & 0 & -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

$$\rho\left(I-A\right) \le \left\|I-A\right\|_{\infty} = \frac{1}{2} < 1$$

So

$$\lim_{k\to\infty}\left(I-A\right)^{k} = 0$$

$$\lim_{k\to\infty}\left[I - \left(I-A\right)^{k}\right] A^{-1}h^{*} = A^{-1}h^{*}$$

Then

$$\lim_{k\to\infty} h^{\langle k\rangle} = A^{-1}h^{*}$$

So

$$\lim_{k\to\infty} h^{*\langle k\rangle} = \lim_{k\to\infty} Ah^{\langle k\rangle} = AA^{-1}h^{*} = h^{*}$$

$$\lim_{k\to\infty}\left\|e^{\langle k\rangle}\right\| = \lim_{k\to\infty}\left\|h^{*} - h^{*\langle k\rangle}\right\| = \lim_{k\to\infty}\left(h^{*} - h^{*\langle k\rangle}\right)^{T}\left(h^{*} - h^{*\langle k\rangle}\right) = 0$$

4.1.2 Proof of theorem 2:

$$\begin{aligned}\left\|e^{\langle k\rangle}\right\| &= \left\|h^{*}-h^{*\langle k\rangle}\right\| \\ &= \left\|h^{*}-\left(I-(I-A)^{\langle k\rangle}\right)h^{*}\right\| \\ &= \left\|(I-A)^{\langle k\rangle} h^{*}\right\| \\ &\geq \left\|(I-A)^{\langle k+1\rangle} h^{*}\right\| \\ &= \left\|e^{\langle k+1\rangle}\right\| \\ &(k=1,2\cdots)\end{aligned}$$

Namely

$$\left\|e^{\langle 1\rangle}\right\| \geq \left\|e^{\langle 2\rangle}\right\| \geq \cdots \left\|e^{\langle k\rangle}\right\| \geq \cdots$$

$\left\|e^{\langle k\rangle}\right\|$ is a gradually decrease sequence.

In fact, according to the proof of the theorm 2,we can know

$$\begin{aligned}\frac{\left\|h^{*\langle k+1\rangle}-h^{*}\right\|}{\left\|h^{*\langle k\rangle}-h^{*}\right\|} &= \frac{\left\|(I-A)(I-A)^{k} h^{*}\right\|}{\left\|(I-A)^{k} h^{*}\right\|} \\ &\leq \frac{\left\|I-A\right\|\left\|(I-A)^{k} h^{*}\right\|}{\left\|(I-A)^{k} h^{*}\right\|} \\ &= \left\|I-A\right\| \\ &\leq \frac{1}{2}\end{aligned}$$

So, the iterative sequence is linear convergence at least, but the numerical experiment shows that the result is satisfactory.

4.2 When control points in general model(etc. equidistant and unequidistant)

4.2.1how to get the parameter

Suppose

$$t_i = \begin{pmatrix} t_i^x \\ t_i^y \\ t_i^z \end{pmatrix} = \begin{pmatrix} \dfrac{h_i^x - h_{i-1}^x}{h_{i+1}^x - h_{i-1}^x} \\ \dfrac{h_i^y - h_{i-1}^y}{h_{i+1}^y - h_{i-1}^y} \\ \dfrac{h_i^z - h_{i-1}^z}{h_{i+1}^z - h_{i-1}^z} \end{pmatrix}$$

$(i = 1,2,\cdots N)$,

For the repeated data, we deal with them as a point. so

$$0 < t_i < 1 \ (i = 1,2,\cdots N)。$$

Modify the algorithm in reference [1] as follows, then we can get the algorithm under the unequidistant condition. In step 1

$$h_i^{*<1>} = \vec{s}_i^{\,<1>}(\frac{1}{2}) \qquad (i=1,2,\cdots N)$$

replace $\frac{1}{2}$ for t_i ,then we can get the algorithm under the unequidistant condition.

4.2.1The iterative matrix under the unequidistant condition

Refer to the front proof process, we can get its iterative matrix is

$$A=\begin{bmatrix} \frac{-2t_N^2+2t_N+1}{2} & \frac{t_N^2}{2} & 0 & 0 & \cdots & \frac{t_N^2-2t_N+1}{2} & 0 & 0 \\ \frac{t_1^2-2t_1+1}{2} & \frac{-2t_1^2+2t_1+1}{2} & \frac{t_1^2}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{t_2^2-2t_2+1}{2} & \frac{-2t_2^2+2t_2+1}{2} & \frac{t_2^2}{2} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{t_N^2-2t_N+1}{2} & \frac{-2t_N^2+2t_N+1}{2} & \frac{t_N^2}{2} \\ 0 & 0 & \frac{t_1^2}{2} & 0 & \cdots & 0 & \frac{t_1^2-2t_1+1}{2} & \frac{-2t_1^2+2t_1+1}{2} \end{bmatrix}$$

for

$$\left|\frac{-2t_i^2+2t_i+1}{2}\right|-\left|\frac{t_i^2-2t_i+1}{2}\right|-\left|\frac{t_i^2}{2}\right|$$
$$=\frac{(-2t_i^2+2t_i+1)-(t_i^2-2t_i+1)-(t_i^2)}{2}$$
$$=-2(t_i-\frac{1}{2})^2+\frac{1}{2}>0$$

$(i=1,2,\cdots N)$

So A is N+2 rank rank strict principal diagonal excellent matrix, so matrix A is inversable.

$$\|I-A\|_\infty \le \max_{1\le i\le N}\left\{\left|1-\frac{-2t_i^2+2t_i+1}{2}\right|+\left|-\frac{t_i^2}{2}\right|+\left|-\frac{t_i^2-2t_i+1}{2}\right|\right\}$$
$$=\max_{1\le i\le N}\left\{\frac{2t_i^2+2t_i+1}{2}+\frac{t_i^2}{2}+\frac{t_i^2-2t_i+1}{2}\right\}$$
$$=\max_{1\le i\le N}\left\{2(t_i-\frac{1}{2})^2+\frac{1}{2}\right\}$$
$$<1$$

Where

$$0<t_i<1 \ (i=1,2,\cdots N)$$

So

$$\rho(I-A)\le\|I-A\|_\infty<1$$

So refer to the proof process of theorem 2 ,3 we can prove theorem 2 ,3 are also correct when the control points are unequidistant.

5. Numerical experiment

5.1 Draw curve according to the disperse data points

Let the design- data points as :

$$\left(\begin{array}{c|ccccccc|c} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ -6.00 & 6.00 & 4.00 & 0.00 & 5.00 & 3.00 & -4.00 & -6.00 & 6.00 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

According to the algorithm provided above, we can get figure 2, figure 3 and table 1

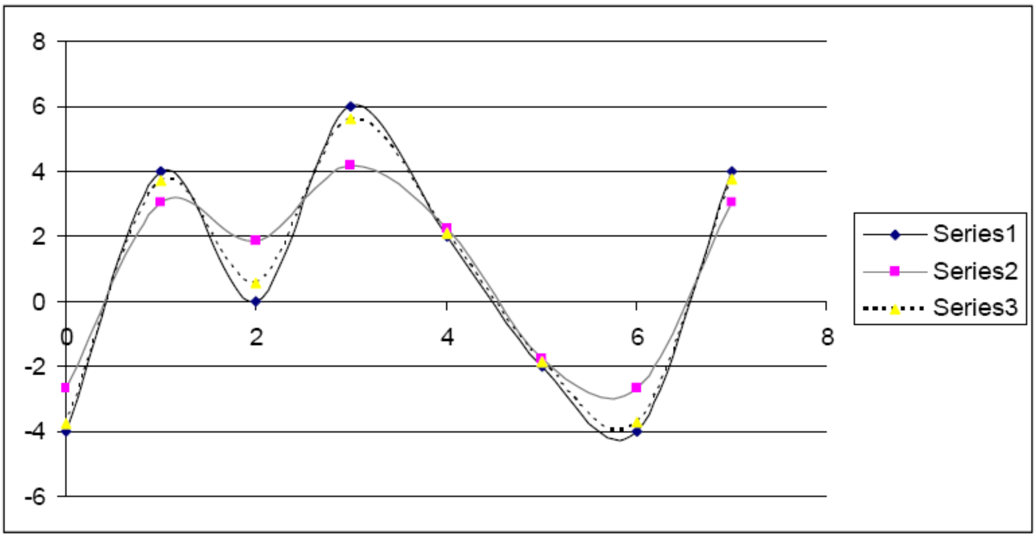

Fig. 2. a numerical example of the periodic interpolated approximate B-spline.

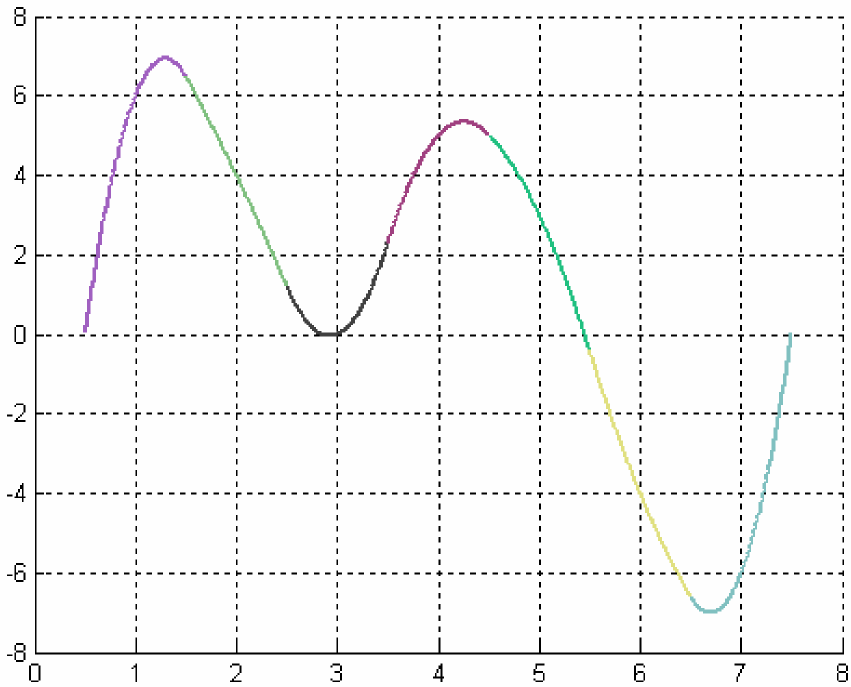

Fig. 3 the result of approach effect sketch map

Table 1 the approach process data variety table(k is the iterative times)

	i=0	i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8
k=1	-4.25	4.25	3.75	1.125	4.125	2.375	-3.375	-4.25	4.25
k=2	-5.4219	5.375	4.0156	0.42188	4.7188	2.875	-3.9844	-5.4219	5.375
k=3	-5.7793	5.7695	4.0293	0.13867	4.8926	3.002	-4.0527	-5.7793	5.7695
k=4	-5.9094	5.9111	4.0188	0.044434	4.9556	3.0205	-4.041	-5.9094	5.9111
k=5	-5.9611	5.9641	4.0103	0.014313	4.9808	3.0158	-4.0241	-5.9611	5.9641
k=6	-5.9828	5.9849	4.0053	0.0046997	4.9914	3.0094	-4.0129	-5.9828	5.9849
k=7	-5.9922	5.9934	4.0026	0.0015888	4.9961	3.005	-4.0065	-5.9922	5.9934
k=8	-5.9964	5.9971	4.0013	0.00055784	4.9982	3.0026	-4.0032	-5.9964	5.9971
k=9	-5.9983	5.9987	4.0006	0.00020481	4.9992	3.0013	-4.0016	-5.9983	5.9987
k=10	-5.9992	5.9994	4.0003	7.8903e-005	4.9996	3.0006	-4.0008	-5.9992	5.9994
k=11	-5.9996	5.9997	4.0001	3.1865e-005	4.9998	3.0003	-4.0004	-5.9996	5.9997
k=12	-5.9998	5.9999	4.0001	1.3422e-005	4.9999	3.0001	-4.0002	-5.9998	5.9999
k=13	-5.9999	5.9999	4	5.8523e-006	5	3.0001	-4.0001	-5.9999	5.9999
k=14	-6	6	4	2.6204e-006	5	3	-4	-6	6
k=15	-6	6	4	1.1962e-006	5	3	-4	-6	6
Control points	-6.00	6.00	4.00	0.00	5.00	3.00	-4.00	-6.00	6.00

5.2 The function approaches experiment

We use

$$y = x^2$$

$$y = \frac{1}{1+x^2}$$

$$y = \sin(x)$$

as the test function .The result are shown in figure 4, 5 and 6:("*" stands for the points in original function, curve is the result of the iterative B Spline curve)

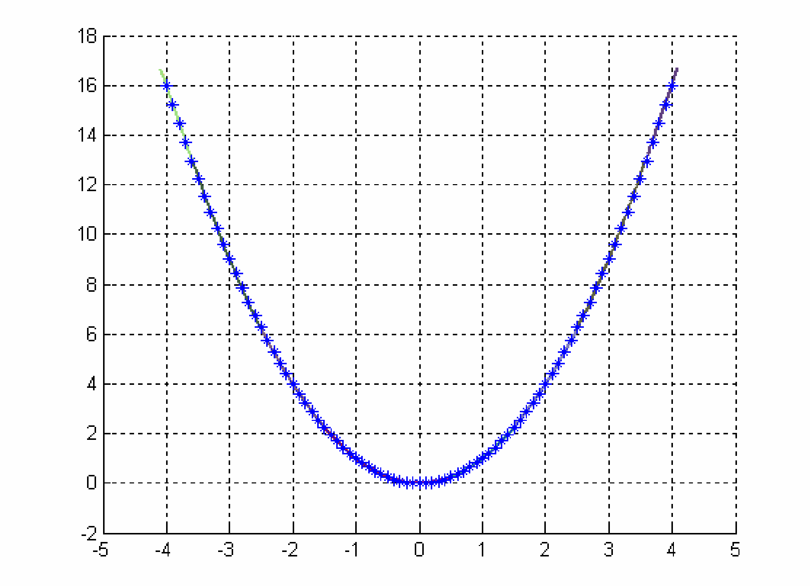

Fig. 4 $y = x^2$ approach effect sketch map

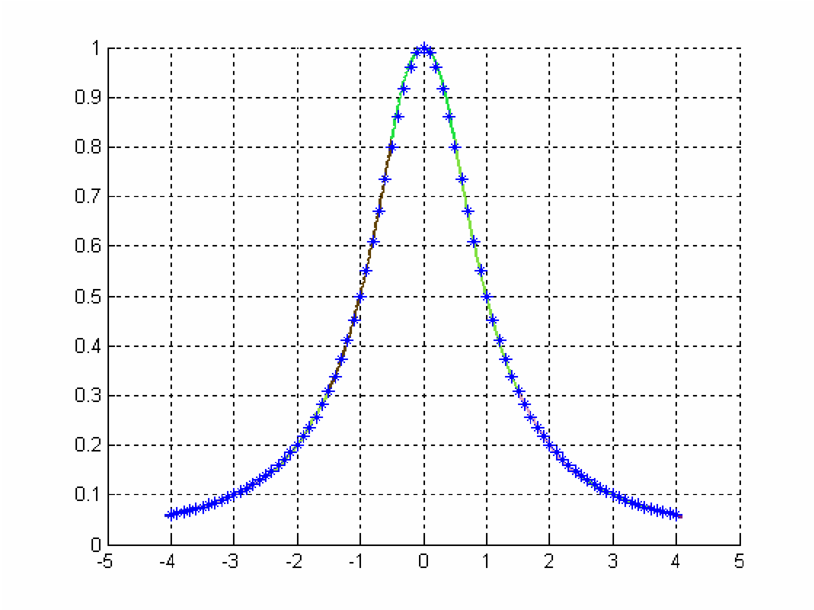

Fig. 5 $y = \frac{1}{1+x^2}$ approach effect sketch map

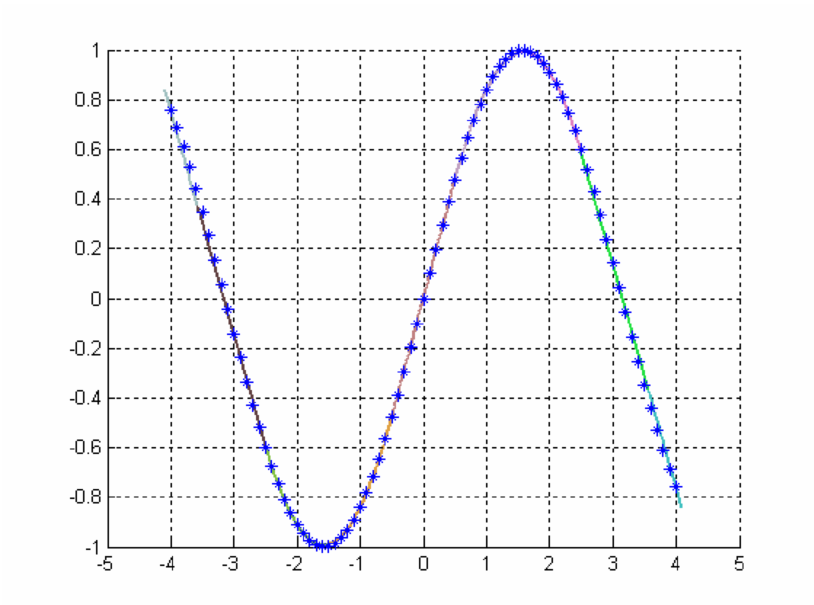

Fig. 6 $y = \sin(x)$ approach effect sketch map

6. Conclusion

The numerical experiment above indicates that iterative algorithm provided in reference [1] is easily and efficiently. This algorithm converges quickly and has high accuracy. We can use this algorithm to draw up disperse data points and approach complicated function etc.

Moreover, reference [1] explains the algorithm based on quadratic B-Spline. In fact, the algorithm can be applied to the Bezier spline curve, rational Bezier spline curve, rational B spline curve, NURBS spline curve and other spline curve.

Yong Jiang and the others applied this algorithm to spinning product design and get very nice result. Shown as figure 7(this figure is chosen from reference[1]), the details can be seen in refenence [1].

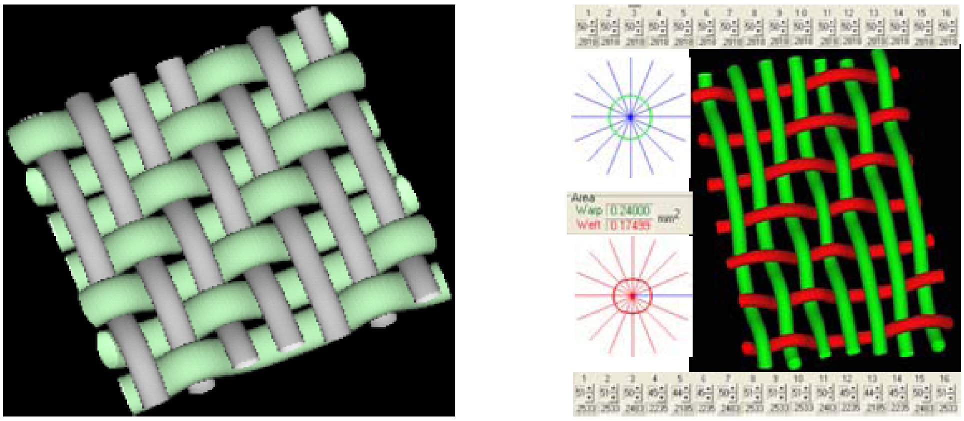

Fig.7 curves approach calculate way in the spinning the design of application Reference

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