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Numerical Solution of Nonlinear Volterra Integral Equations of the Second Kind by Power Series

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Abstract. In this paper, we present a recursive method for solving nonlinear Volterra integral equations. The proposed method obtains Taylor expansion of the exact solution of Volterra integral equation by using simple computations. Comparison with other methods proves that the proposed method is very effective and convenient.

Keywords: Nonlinear Volterra integral equation of second kind, Numerical method, Taylor expansion.

1. Introduction

Several numerical approaches for approximating the solution of linear or nonlinear Volterra integral equations are known. Tricomi [5], introduced the classical method of successive approximations for nonlinear Volterra integral equations.Brunner [1] applied a collocation-type method to nonlinear Volterra equations and integro-differential equations and discussed its connection with the iterated collocation method. For Volterra-Hammerstien equations, the asymptotic error expansion of a collocation method was introduced [2]. In general, most of numerical methods transform the integral equation to a linear or nonlinear system of algebraic equations which can be solved by direct or iterative methods. Yousefi and Razzaghi [6], and also Maleknejad et al. [3] used Legendre wavelets to numerical solution of linear and nonlinear Volterra integral equations. Recently,Chebyshev polynomials are applied for solving of nonlinear Volterra integral equations of the second kind of the form [4]:

$$y(s) = g(s) + \int_a^s K(s,t)\left[y(t)\right]^p dt. \tag{1}$$

In equation (1), the functions $g(s)$ and $K(s,t)$ are known, and $y(s)$ is the unknown function to be determined, also $p \geq 1$ is a positive integer number.

For $p=2$, Eq. (1) is a nonlinear Hammertein-type integral equation.

Without loss generality, we assume that in Eq.(1), $a=0$, because in the other wise with substituting $s=z-a$ in Eq.(1) and then doing change variable $t=x-a$, we have the following integral equation,

$$y(z-a) = g(z-a) + \int_0^z K(z-a, x-a)\left[y(x-a)\right]^p dx.$$

If we set $s=0$ in Eq.(1) then we have $y(0)=g(0)$, as the initial condition.

Hence, the solution of Eq.(1) can be assume that,

$$y = e_0 + e_1 s, \tag{2}$$

where, e_1 is a unknown parameter and,

$$e_0 = g(0). \tag{3}$$

Substitute Eq.(2) into Eq.(1) and neglect higher order terms, we have linear equation of e_1 in the form,

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$$ae_1 = b, \tag{4}$$

where, a and b are known constants, solve Eq.(4), the coefficients of s in Eq.(2) can be determined. Repeating above procedure for higher order terms, we can get e arbitrary order power series of the solution for Eq.(1).

2. The Method

Suppose the solution of Eq.(1) with $e_0 = y(0) = g(0)$ as the initial condition to be as follows,

$$y(x) = e_0 + e_1 s, \tag{5}$$

where, e_1 is a unknown parameter.

If we substitute Eq.(5) into Eq.(1) linear algebraic equation:

$$ae_1 - b + Q(s) = 0, \tag{6}$$

where, a and b are known constant values and $Q(s)$ is a polynomial with the order greater than zero.

By neglecting $Q(s)$ in (6) and solving the system of $ae_1 = b$, the unknown parameter e and therefore the coefficient of s in Eq.(5) obtains.

In the next step, we assume that the solution of Eq.(1) to be,

$$y(s) = e_0 + e_1 s + e_2 s^2, \tag{7}$$

here, e_0 and e_1 both are known and e_2 is unknown parameter.

By substituting Eq.(7) into Eq.(1), we have following system,

$$\left(ae_2 - b\right) s + Q(s^2) = 0. \tag{8}$$

By neglecting and solving the system of $ae_2 = b$, the unknown parameter e_2 and therefore the coefficient of s^2 in Eq.(7) obtains.

By repeating the above procedure for m iteration, a power series of the following form derives,

$$y(s) = e_0 + e_1 s + e_2 s^2 + \cdots + e_m s^m. \tag{9}$$

Equation (9) is an approximation for the exact solution $y(s)$ of the integral equation (1).

Theorem 2.1. 1 *Let $y = f(s)$ be the exact solution of the following Volterra integral equation*

$$y(s) = g(s) + \int_a^s K(s,t)\left[y(t)\right]^p dt. \tag{10}$$

Then, the proposed method obtains the Taylor expansion of $f(s)$.

Proof. Without loss of generality, we suppose that $a = 0$. As it was showed, in the presented method, we assume that the approximate solution to Eq.(10) be as follows,

$$\widetilde{f}(s) = e_0 + e_1 s + e_2 s^2 + \cdots, \tag{11}$$

Hence, it is sufficient that we only prove,

$$e_l = \frac{f^{(l)}(0)}{l!}, \qquad l = 1,2,3,\ldots. \tag{12}$$

Note that for $l = 0$, we always have

$$f(0) = g(0) = e_0. \tag{13}$$

For $l = 1$, if we set $y = f(s)$ and then derivative from Eq.(10), we obtain

$$f'(s) = g'(s) + K(s,s)\left[f(s)\right]^p \tag{14}$$

setting $s = 0$ in (14), then we get

$$f'(0) = g'(0) + K(0,0)\left[f(0)\right]^p. \tag{15}$$

In other hand, from (11) and (13), we have

$$\tilde{f}(s) = e_0 + e_1 s \tag{16}$$

by substituting (16) into (14) and setting $s=0$, we get

$$e_1 = g'(0) + K(0,0)(e_0)^p = g'(0) + K(0,0)(f(0))^p \tag{17}$$

therefore, with comparison (15) and (17), we conclude that,

$$e_1 = f'(0). \tag{18}$$

For $l=2$, similar to the last step, this time we derivative from (14), we have

$$f''(s) = g''(s) + K'(s,s)\left[f(s)\right]^p + pK(s,s)f'(s)\left[f(s)\right]^{p-1} \tag{19}$$

again, we set $s=0$ in (19) and we get

$$f''(0) = g''(0) + K'(0,0)\left[f(0)\right]^p + pK(0,0)f'(0)\left[f(0)\right]^{p-1} \tag{20}$$

According to (11),(13) and (18), we suppose that

$$\tilde{f}(s) = f(0) + f'(0)s + e_2 s^2, \tag{21}$$

by substituting (21) into (19), and setting $s=0$, we have

$$2e_2 = g''(0) + K'(0,0)\left[f(0)\right]^p + pK(0,0)f'(0)\left[f(0)\right]^{p-1}. \tag{22}$$

So, with comparison (20) and (22), we conclude that

$$2e_2 = f''(0), \quad \text{or} \quad e_2 = \frac{f''(0)}{2!}.$$

By continuing the above procedure, we can easily prove (12) for $l = 3,4,\ldots$.

Corollary 2.2. 2 *If the exact solution to Eq.(10) be a polynomial, then the proposed method will obtain the real solution.*

3. Applications

Example 3.1. 3 *The test problem, consider the following nonlinear integral equation*

$$y(s) = g(s) + \int_0^s (s+t)[y(t)]^2 dt, \tag{23}$$

where, $g(s) = e^s - se^{2s} + \frac{e^{2s}}{4} + \frac{s}{4} - \frac{1}{4}$ and $y(s) = e^s$ is the exact solution.

If we set $s=0$ in Eq.(23), we get $y(0)=1$ as the initial condition to Eq.(23).

Here, we need obtain the Taylor expansion of $g(s)$ and substitute it in Eq.(23).

Now, we apply the proposed method for solving Eq.(23).

Let the solution of Eq.(23) to be,

$$y(s) = y(0) + e_1 s \Rightarrow y(s) = 1 + e_1 s \tag{24}$$

Substitute (24) into Eq.(23), we have,

$$1 + e_1 s = e^s - se^{2s} + \frac{e^{2s}}{4} + \frac{s}{4} - \frac{1}{4} + \int_0^s (s+t)(1+e_1 s)^2 dt$$

After simplifying, we get,

$$(e-1)s + Q(s^2) = 0 \tag{25}$$

where $Q(s^2) = \frac{83}{1008}s^7 + \frac{35}{144}s^6 + \frac{71}{120}s^5 - \left(\frac{7}{12}e_1^2 + \frac{9}{8}\right)s^4 - \left(\frac{5}{3}e_1 - \frac{3}{2}\right)s^3 + \frac{1}{2}s^2.$

By neglecting $Q(s^2)$ in (25) and solving equation $e-1=0$, we obtain $e=1$, hence

$$y(s) = 1 + s, \tag{26}$$

is the first approximation for the exact solution to Eq.(23).

Now, fom (26), the solution of Eq.(23) can be supposed as:

$$y(s) = 1 + s + e_2 s^2 \tag{27}$$

Substituting (27) into Eq.(23), gives,

$$(e_2 - \frac{1}{2})s^2 + Q(s^3) = 0, \tag{28}$$

whre,

$$Q(s^3) = \frac{83}{1008}s^7 - \left(\frac{11}{30}e_2^2 - \frac{35}{144}\right)s^6 - \left(-\frac{71}{120} + \frac{9}{10}e_2\right)s^5 - \left(-\frac{13}{24} + \frac{7}{6}e_2\right)s^4 - \frac{1}{6}s^3.$$

By neglecting $Q(s^3)$ in (28), we have $e_2 = \frac{1}{2}$.

Putting $e = \frac{1}{2}$ in (27), we get

$$y(s) = 1 + s + \frac{1}{2}s^2 \tag{29}$$

From (29) the solution of Eq.(23) can be supposed as

$$y(s) = 1 + s + \frac{1}{2}s^2 + e_3 s^3 \tag{30}$$

Substituting (30) into Eq.(23), we obtain

$$(e_3 - \frac{1}{24})s^3 + Q(s^4) = 0, \tag{31}$$

where,

$$Q(s^4) = -\frac{15}{56}e_3^2 s^8 - \left(\frac{13}{42}e_3 - \frac{83}{1008}\right)s^7 - \left(-\frac{109}{720} + \frac{11}{15}e_3\right)s^6 - \left(-\frac{17}{120} + \frac{9}{10}e_3\right)s^5 - \frac{1}{24}s^4.$$

By ignoring $Q(s^4)$ in (31), we obtain $e = \frac{1}{24}$. Therefore

$$y(s) = 1 + s + \frac{1}{2}s^2 + \frac{1}{24}s^3. \tag{32}$$

Proceeding in this way, we get,

$$y(s) = 1 + s + \frac{1}{2}, s^2 + \frac{1}{6}s^3 + \frac{1}{24}s^4 + \frac{1}{120}s^5 + \frac{1}{720}s^6 + \frac{1}{5040}s^7 + \cdots. \tag{33}$$

Example 3.2. 4 As the second example consider the following linear Volterra integral equation with logarithmic singularity,

$$y(s) = g(s) + \int_0^s \ln|s - t + 1|\, y^2(t)dt, \quad -1 \le t \le s \le 1 \tag{34}$$

where, $g(s) = \left(\frac{3}{4} - \frac{1}{2}\ln(s+1)\right)s^2 + \left(-2\ln(s+1) + \frac{3}{2}\right)s + \sqrt{s} - \frac{3}{2}\ln(s+1)$ and the exact solution is $Y(s) = \sqrt{s+1}$.

By applying the proposed method, we obtain,

$$y(s) = 1 + \frac{1}{2}s - \frac{1}{8}s^2 + \frac{1}{16}s^3 - \frac{5}{128}s^4 + \frac{7}{256}s^5 - \frac{21}{1024}s^6 + \cdots$$

Table 1 shows the absolute errors for $m = 6$.

Table 1. Absolute errors for Example 3.2.

s_i	$y(s_i)$	$Y(s_i)$	*Error*
-1	0.22558593750	0	$2.2558593750\times10^{-1}$
-0.75	0.50582194328	0.50000000000	$5.8219432831\times10^{-3}$
-0.50	0.70732116699	0.70710678119	$2.1438580564\times10^{-4}$
-0.25	0.86602663994	0.86602540378	$1.2361539159\times10^{-6}$
0	1.0000000000	1.0000000000	0
0.25	1.1180331707	1.1180339887	$8.1804982166\times10^{-7}$
0.50	1.2246551514	1.2247448714	$8.9720024401\times10^{-5}$
0.75	1.3215339184	1.3228756555	$1.3417371515\times10^{-3}$
1	1.4052734375	1.4142135624	$8.9401248731\times10^{-3}$

Example 3.3. 5 For the following nonlinear second kind Volterra integral equation

$$y(s)=3s-1-\frac{1}{3}e^{s}\left(13-10e^{s+1}+12s+9s^{2}\right)+\int_{-1}^{s}\left(-\frac{1}{3}\right)e^{2s-t}\left[y(t)\right]^{2}dt$$

with exact solution $y(s)=3s-1$, the method gives the exact solution too, because it is a polynomial.

Example 3.4. 6 *We consider the nonlinear Volterra integral equation*

$$u(x)=f(x)+\int_{0}^{x}F(x,\xi)N_{1}(u(\xi))d\xi,\quad x\in[0,1]\quad ,u(x),f(x)\in W_{2}^{1}\in[0,1]. \qquad (35)$$

the existence and uniqueness of solution for Eq.(35) have to be established in [7,8].

$$N_1(u(x))=1+u^2(x),$$
$$F(x,\xi)=sin(x-\xi),$$
$$f(x)=\frac{-3}{2}+\frac{1}{6}cos2x+\frac{7}{3}cosx$$

then $u(x)=cosx$ is the exact solution.

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