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The Nonclassical Symmetries and Group Invariant Solutions of the Boussinesq-Burgers Equation

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Abstract. In this paper, the nonclassical symmetries and group invariant solutions of the Boussinesq-Burgers equation have been discussed. By using the nonclassical method, we obtain nonclassical symmetries that reduce the Boussinesq-Burgers equation to ordinary differential equation, and several invariant solutions. We remark that some of them are new solutions of the Boussinesq-Burgers equation.

Keywords: Boussinesq-Burgers equation; nonclassical symmetries; determining equation; group-invariant solutions

1. Introduction

The symmetry analysis has played an important role in the construction of exact solutions to nonlinear partial differential equations. The nonclassical symmetries method (NSM) was introduced in 1969 by Bluman and Cole [1] in order to obtain new exact solutions of the linear heat equation. The NSM consists of adding the invariant surface condition to the given equation, and applying the classical symmetries method (CSM) [2]. Besides, the NSM may yield more solutions than the CSM. The NSM has been successfully applied to various equations, for the purpose of finding new exact solutions [3, 4]. Presently there exists an extensive body of literature in which we refer the reader to the books by Bluman and Kumei [5] Olver[6] and Rogers and Ames[7].

This paper is arranged as follows: we first introduce differences between CSM and NSM briefly.

In section 2, we introduce the NSM by Solving the Burgers equation. In section 3, we make use of the NSM to obtain the nonclassical symmetries of the Boussinesq-Burgers equation. In section 4, we obtain the group invariant solutions of the Boussinesq-Burgers equation. In section 5, we make some final comments.

For equations in (1+1) dimensions, one seeks the invariance of a differential equation:

$$\Delta_1(x,t,u,u_t,u_x,u_{tt},u_{xt},...)=0 \quad . \tag{1.1}$$

Suppose the form of Eq. (1.1) is invariant under a group action on (x,t,u) space given by its infinitesimals [8]:

$$x^* = x + X(x,t,u)\varepsilon + o(\varepsilon^2),$$
$$t^* = t + T(x,t,u)\varepsilon + o(\varepsilon^2),$$
$$u^* = u + U(x,t,u)\varepsilon + o(\varepsilon^2).$$

The invariance requirement is

$$\Gamma^{(n)}\Delta_1 /_{\Delta_1=0} = 0, \tag{1.2}$$

where $\Gamma^{(n)}$ is the n th extension of infinitesimal generator[9]

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$$\Gamma = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u} \tag{1.3}$$

This leads to a set of determining equations for the infinitesimals *X*, *T* and *U* that, when solved, gives rise to the symmetries of (1.1). Once symmetries are known for a differential equation, invariance of the solution leads to the invariant surface condition [10]:

$$\Delta_2 = Xu_x + Tu_t - U = 0\,. \tag{1.4}$$

Solutions of (1.4) lead to a solution Ansatz, which, when substituted into Eq.(1.1) gives a reduction of the original equation. A generalization of the so-called "classical method" of Lie was proposed by Bluman and Cole. Today, it is commonly referred to as the "nonclassical method". Their method seeks invariance of the original equation augmented with the invariant surface condition (1.4).

However, all exact solutions obtained by the classical method also be obtained by the nonclassical method .Unlike the determining equations for the classical method which are linear, the determining equations for the nonclassical method are usually highly non-linear.

$$\Gamma^{(n)}\Delta /_{\Delta_1=0,\Delta_2=0} = 0\,. \tag{1.5}$$

Solving this governing equation leads a set of determining equations for infinitesimals *X*, *T* and *U*. Solving the determining equations gives rise to the nonclassical symmetries of Eq. (1.1)., These symmet- ries gives a reduction of the original equation. When the reduced equation is solved, we obtain the invariant solutions under group of the original equation (1.1).

2. The nonclassical symmetries and group invariant solutions of the Burgers equation

In this section, we make use of a simple equation to introduce the NSM. In 1948, Johannes Martinus Burgers put forward in the paper *A Mathematical Model Illustrating the Theory of Turbulence* a type of equations called the Burgers equation, which was used to describe the turbulence of the free fluid. The Burgers equation is expressed as follows:

$$u_t + uu_x - u_{xx} = 0 \tag{2.1}$$

If we denote the Burgers Eq.(2.1)by Δ_1 and the invariant surface condition with *T*=*k* by Δ_2 then

$$\Delta_1 = u_t + uu_x - u_{xx} \tag{2.2}$$

$$\Delta_2 = Tu_t + Xu_x - U \tag{2.3}$$

The determining equations for the nonclassical symmetries of the wave equation are obtained by requiring the governing equation as follows:

$$\Gamma^{(2)}\Delta_1 /_{\Delta=0_1,\Delta_2=0} = 0 \tag{2.4}$$

Where the infinitesimal generator Γ is given by

$$\Gamma = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u}$$

With the second extensions as

$$\Gamma^{(1)} = \Gamma + U_{[t]}\frac{\partial}{\partial u_t} + U_{[x]}\frac{\partial}{\partial u_x}$$

$$\Gamma^{(2)} = \Gamma^{(1)} + U_{[tt]}\frac{\partial}{\partial u_{tt}} + U_{[tx]}\frac{\partial}{\partial u_{tx}} + U_{xx}\frac{\partial}{\partial u_{xx}} \tag{2.5}$$

The coefficients of the operators in (2.5) is given by

$$U_{[t]} = D_t(U - Xu_x - Tu_t) + Xu_{tx} + Tu_{tt} = D_t(U - Xu_x) + Xu_{tx} \tag{2.6}$$

$$U_{[x]} = D_x(U - Xu_x - Tu_t) + Xu_{xx} + Tu_{xt} = D_x(U - Xu_x) + Xu_{xx} \tag{2.7}$$

$$U_{[xx]} = D_{xx}(U - Xu_x - Tu_t) + Xu_{xxx} + Tu_{xxt} = D_{xx}(U - Xu_x) + Xu_{xxx} \tag{2.8}$$

Invariance of the nonlinear wave equation is given by Eq.(2.4),which by (2.5) gives

$$\Gamma^{(2)}\Delta_1 /_{\Delta_1=0,\Delta_2=0} = U_{[t]} + Uu_x + uU_{[x]} - U_{[xx]} = 0 \tag{2.9}$$

Substituting (2.6)-(2.8)into (2.9) gives the governing equation for the infinitesimals *X*,*T*,*U*. Solving this governing equation leads to a set of the determining equations for *X*,*T*,*U*. Consequently, solving determining equations, we can obtain the nonclassical symmetries are:

$$\sigma_1 = 1 - tu_x$$

$$\sigma_2 = u_x \ \sigma_3 = u_t$$

Next, we will make use of these symmetries to obtain the group invariant solutions of Burgers equation.

Lemma. If σ is one of the symmetries of the equation (1.1), then the group invariant solutions for the corresponding invariant group can be obtained by solving the following equations:

$$\begin{cases} \Delta_1(x,t,u,v,u_x,u_t,v_x,v_t...) = 0 \\ \sigma(u) = 0. \end{cases}.$$

As is known to us all, if σ_1, σ_2 and σ_3 are symmetries of an equation, their linear displacements are still symmetries of the equation.

Case 1: $\sigma_4 = \sigma_3 + \sigma_1$. Following $u_t - tu_x + 1 = 0$ and solving this partial differential equation, we can obtain $\zeta = x + \frac{t^2}{2}$, $u = f(\zeta) - t$. Substituting it into (2.1), we can get a second-order ordinary differential equation:

$$f'' - ff' + 1 = 0.$$

Solving the equation above, under initial condition $f(\zeta)/_{\zeta=0} = 0$, we can obtain a special solution: $f(\zeta) = -c_1\zeta - \frac{1}{2}\zeta^2 + \frac{1}{6}c_1^2\zeta^3 + \frac{1}{8}c_1\zeta^4 + ...$

So $u_1(x,t) = -c_1(x+\frac{t^2}{2}) - \frac{1}{2}(x+\frac{t^2}{2})^2 + \frac{1}{6}c_1^2(x+\frac{t^2}{2})^3 + \frac{1}{8}c_1(x+\frac{t^2}{2})^4 + ... - t$ is the group invariant solution of the Burgers equation.

Case 2: $\sigma_5 = \sigma_1 + \sigma_2$. Following $1 - tu_x + u_x = 0$ and solving this equation, we can obtain $u_2(x,t) = \frac{x}{t-1} + \varphi(t)$ .Substituting it into (2.1).we can get $\varphi(t) = \frac{c}{t-1}$.

Therefore, $u_2(x,t) = \frac{x+c}{t-1}$ is the group invariant solution of the Burgers equation, where c is a constant.

Case 3: $\sigma_6 = \sigma_2 - \sigma_3$. Following $u_x - u_t = 0$ and solving this equation, we can get $\zeta = x + t$,$u = f(\zeta)$ .Substituting them into (2.1), we can get a second-order differential equation:

$$f'' - ff' - f' = 0.$$

Solving this equation, we can obtain $f(\zeta) = \sqrt{2c_1 - 1}\tan[\sqrt{2c_1 - 1}(\frac{1}{2}\zeta + c_2)] - 1$.

So $u_3(x,t) = \sqrt{2c_1 - 1}\tan[\sqrt{2c_1 - 1}(\frac{1}{2}\zeta + c_2)] - 1$ is the group invariant solution of the Burgers equation, where c_1 and c_2 are constants and $\zeta = x + t$.

Case 4: $\sigma_7 = \sigma_1 + \sigma_2 + \sigma_3$. Following $1 - tu_x + u_x + u_t = 0$ and solving this partial differential equation, we can obtain $\zeta = t - \frac{t^2}{2} - x$, $u = f(\zeta) - t$. Substituting them into (2.1), we can get a differential equation: $f'' + ff' - f' + 1 = 0$.

If its solution $f(\zeta)$ exist, and then $u = f(\zeta) - t$ is the group invariant solution of the original equation.

3. The nonclassical symmetries for the Boussinesq- Burgers equation

It is well known that the nonlinear partial differential equations (NPDEs) are widely used to describe many important phenomena in physics, biology, chemistry, etc. In the past several decades, a great number of efforts have been made to study various nonlinear wave equations. There are many ways to obtain exact solutions of partial differential equations, such as the Darboux transformation [11], the algebra-geometric method, the Inverse Scattering transformation, and so on. In this paper, nonclassical symmetries method was used to solve Boussinesq-Burgers equation.

The Boussinesq-Burgers equation is expressed as follows:

$$\begin{cases} u_t = -2uu_x + \frac{1}{2}v_x \\ v_t = \frac{1}{2}u_{xxx} - 2(uv)_x \end{cases} . \qquad \textbf{(3.1)}$$

If we denote Eq.(3.1) by Δ_1 and Δ_1' and the invariant surface condition with $T_1 = \frac{1}{k}$ and $T_2 = \frac{1}{h}$ ($k \neq 0, h \neq 0$) by Δ_2 , Δ_2' ,then

$$\Delta_1 = u_t + 2uu_x - \frac{1}{2}v_x \ , \qquad (3.2)$$

$$\Delta_1' = v_t - \frac{1}{2}u_{xxx} + 2(uv)_x \qquad (3.3)$$

$$\Delta_2 = T_1 u_t + X_1 u_x - U \ , \qquad (3.4)$$

$$\Delta_2' = T_2 v_t + X_2 v_x - V \qquad (3.5)$$

The determining equations for the nonclassical symmetries of the Boussinesq-Burgers equation are obtained by requiring the governing equation as follows:

$$\begin{cases} \Gamma_1^{(1)} \Delta_1 /_{\Delta_1 = 0, \Delta_2 = 0} = 0 \\ \Gamma_2^{(3)} \Delta_1' /_{\Delta_1' = 0, \Delta_2' = 0} = 0 \end{cases} , \qquad (3.6)$$

where the infinitesimal generator Γ_1, Γ_2 is given by

$$\begin{cases} \Gamma_1 = X_1 \frac{\partial}{\partial x} + T_1 \frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial u} + V_1 \frac{\partial}{\partial v} \\ \Gamma_2 = X_2 \frac{\partial}{\partial x} + T_2 \frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial u} + V_2 \frac{\partial}{\partial v} \end{cases} ,$$

with the second extensions as

$$\Gamma_i^{(1)} = \Gamma_i + U_{i[t]} \frac{\partial}{\partial u_t} + U_{i[x]} \frac{\partial}{\partial u_x} + V_{i[t]} \frac{\partial}{\partial v_t} + V_{i[x]} \frac{\partial}{\partial v_x} ,$$

$$\Gamma_i^{(2)} = \Gamma_i^{(1)} + U_{i[tt]} \frac{\partial}{\partial u_{tt}} + U_{i[tx]} \frac{\partial}{\partial u_{tx}} + U_{i[xx]} \frac{\partial}{\partial u_{xx}} + V_{i[tt]} \frac{\partial}{\partial v_{tt}} + V_{i[xt]} \frac{\partial}{\partial v_{xt}} + V_{i[xx]} \frac{\partial}{\partial v_{xx}} ,$$

$$\Gamma_i^{(3)} = \Gamma_i^{(2)} + U_{i[ttt]} \frac{\partial}{\partial u_{xxx}} + U_{i[ttx]} \frac{\partial}{\partial u_{ttx}} + U_{i[txx]} \frac{\partial}{\partial u_{txx}} + U_{i[xxx]} \frac{\partial}{\partial u_{xxx}} + V_{i[ttt]} \frac{\partial}{\partial v_{ttt}}$$

$$+V_{i[ttx]}\frac{\partial}{\partial v_{ttx}}+V_{i[txx]}\frac{\partial}{\partial v_{txx}}+V_{i[xxx]}\frac{\partial}{\partial v_{xxx}}\ . \qquad (i=1,2) \qquad (3.7)$$

The coefficients **of the operators in (3.7) are given by**

$$U_{i[t]}=D_t(U_i-X_1u_x-T_1u_t)+X_1u_{xt}+T_1u_{tt}, \qquad (3.8)$$

$$U_{i[x]}=D_x(U_i-X_1u_x-T_1u_t)+X_1u_{xx}+T_1u_{xt}, \qquad (3.9)$$

$$U_{i[xx]}=D_{xx}(U_i-X_1u_x-T_1u_t)+X_1u_{xxx}+T_1u_{xxt}, \qquad (3.10)$$

$$U_{i[xxx]}=D_{xxx}(U_i-X_1u_x-T_1u_t)+X_1u_{xxxx}+T_1u_{xxxt}, \qquad (3.11)$$

$$V_{i[t]}=D_t(V_i-X_2v_x-T_2v_t)+X_2v_{xt}+T_2v_{tt}, \qquad (3.12)$$

$$V_{i[x]}=D_x(V_i-X_2v_x-T_2v_t)+X_2v_{xx}+T_2v_{xt} \quad (i=1,2). \qquad (3.13)$$

The invariance of the nonlinear partial differential equation is given by Eq. (3.6), which by (3.7) gives

$$\begin{cases}\Gamma_1^{(1)}\Delta_1/_{\Delta_1=0,\Delta_2=0}=U_{1[t]}+2U_1u_x+2uU_{1[x]}-\frac{1}{2}V_{1[x]}=0\\ \Gamma_2^{(3)}\Delta_1'/_{\Delta_1'=0,\Delta_2'=0}=V_{2[t]}-\frac{1}{2}U_{2[xxx]}+2U_{2[x]}v+2uV_{2[x]}=0\end{cases}. \qquad (3.14)$$

Substituting (3.4),(3.5) and (3.8)-(3.13) into (3.14) gives the governing equation for the infinitesimals *X*,*T*,*U*. Solving this governing equation leads to a set of determining equations for *X*,*T*,*U*. The determining equations for the nonclassical symmetries of the Boussinesq-Burgers equation are:

$$-kU_{1u}X_1-X_{1t}-kX_{1u}U_1+2U_1=0,$$

$$kX_{1u}X_1=0,$$

$$2U_{1x}=0,$$

$$2U_{1u}-2X_{1x}=0,$$

$$-2X_{1u}=0,$$

$$-0.5V_{1u}+0.5X_{2x}=0,$$

$$0.5X_{2v}=0,$$

$$U_{1t}+kU_{1u}U_1-0.5V_{1x}=0,$$

$$-hV_{2v}X_2-X_{2t}-hX_{2v}V_2=0,$$

$$hX_{2v}X_2=0,$$

$$-1.5U_{2xxu}+0.5X_{1xxx}=0,$$

$$-1.5U_{2uu}+3X_{1xu}+1.5X_{1ux}=0,$$

$$3X_{1x}-U_{2u}=0,$$

$$1.5X_{1u}=0,$$

$$2U_{2x}=0,$$

$$U_{2u}-X_{1x}=0,$$

$$V_{2x}=0,$$

$$2V_{2v}-2X_{2x}=0,$$

$$-2X_{2v}=0,$$

$$V_{2t} + hV_{2v}V_2 - 0.5U_{2xxx} = 0 .$$

Solving above equations, we can obtain:

$$\begin{cases} X_1 = 2c_1 t + c_2, T_1 = \dfrac{1}{k}, U_1 = c_1, V_1 = 0 \\ X_2 = c_4, T_2 = \dfrac{1}{h}, U_2 = 0, V_2 = c_3 \end{cases},$$

where c_1,c_2,c_3 and c_4 are arbitrary constants.

$$\sigma_{1u} = u_t, \quad \sigma_{2u} = u_x, \quad \sigma_{3u} = u_t + u_x, \quad \sigma_{4u} = 2tu_x - 1, \quad \sigma_{5u} = (2t+1)u_x - 1$$

$$\sigma_{6u} = u_t + 2tu_x - 1, \quad \sigma_{7u} = u_t + (2t+1)u_x - 1$$

$$\sigma_{1v} = v_t + v_x, \quad \sigma_{2v} = v_t, \quad \sigma_{3v} = v_x - 1, \quad \sigma_{4v} = v_x, \quad \sigma_{5v} = v_t - 1, \quad \sigma_{6v} = v_t + v_x - 1.$$

4. The group invariant solutions of the Boussinesq-Burgers equation.

Next, we will make use of these symmetries to obtain the group invariant solutions of Boussinesq-Burgers equation.

Case 1: $\begin{cases} \sigma_{2u} = u_x \\ \sigma_{4v} = v_x \end{cases}$. Following $\begin{cases} u_x = 0 \\ v_x = 0 \end{cases}$, we can get $u = f(t)$, $v = g(t)$. Substituting it into (3.1) yields a first-order ordinary differential equation. Solving it, we can obtain $u = f(t) = A$, $v = g(t) = B$, where A and B are arbitrary constants.

Case 2: $\begin{cases} \sigma_{3u} = u_t + u_x \\ \sigma_{1v} = v_t + v_x \end{cases}$. Following $\begin{cases} u_t + u_x = 0 \\ v_t + v_x = 0 \end{cases}$, solving it, we can obtain $\zeta = x - t$, $u = f(\zeta)$, $v = g(\zeta)$.

Substituting it into (3.1), we can obtain $\begin{cases} -f' = -2ff' + 0.5g' \\ -g' = 0.5f''' - 2f'g - 2fg' \end{cases}$. Integrating it, we can obtain

$0.5f'' - 4f^3 + 6f^2 - 2f + c_5 = 0$. If its solutions $f(\zeta)$ exist, then the group invariant solution of the Boussinesq-Burgers equation is $u = f(\zeta)$.

Case 3: $\begin{cases} \sigma_{4u} = 2tu_x - 1 \\ \sigma_{4v} = v_x \end{cases}$. Following $\begin{cases} 2tu_x - 1 = 0 \\ v_x = 0 \end{cases}$, substituting it into (3.1), we can get $u = \dfrac{x}{2t} + \dfrac{c_6}{t}$,

$v = \dfrac{c_7}{t}$ which is group invariant solution of the Boussinesq-Burgers equation, where c_6 and c_7 is an integral constant.

Case 4: $\begin{cases} \sigma_{5u} = (2t+1)u_x - 1 \\ \sigma_{4v} = v_x \end{cases}$. Following $\begin{cases} (2t+1)u_x - 1 = 0 \\ v_x = 0 \end{cases}$, substituting it into (3.1), we can obtain

$u = \dfrac{x + c_8}{2t+1}, v = \dfrac{c_9}{2t+1}$ which is the group invariant solution of the Boussinesq-Burgers equation.

Case 5: $\begin{cases} \sigma_{1u} = u_t \\ \sigma_{2v} = v_t \end{cases}$. Following $\begin{cases} u_t = 0 \\ v_t = 0 \end{cases}$, we denote $u = f(x)$, $v = g(x)$. Substituting it into (3.1), we can get $\begin{cases} -2ff' + 0.5g' = 0 \\ 0.5f''' - 2f'g - 2fg' = 0 \end{cases}$. Integrating it, we can obtain $0.5f'' - 4f^3 + c_{10} = 0$. If its solutions $f(x)$ exist, and then $u = f(x)$ is the group invariant solution of the original equation (3.1), where c_{10} is a constant.

5. Conclusion

In this paper, we have discussed the Boussinesq-Burgers equation to demonstrate how to obtain the nonclassical symmetries. During the process, we have discussed how the determining equations for the nonclassical symmetries can be derived from the original equation and the invariant surface condition. Besides, by using symmetries, we can reduce the partial differential equation to an ordinary differential equation. Consequently, some of group-invariant solutions we obtain have not been found in other papers.

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