ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 4, No. 1, 2009, *pp*. 017-025

Operator Smoothing Splines and the Smoothing Appoximate Solutions of Operate Equaions in Hilbert Spaces

Xinjian Zhang +

College of Science, National University of Defense Technology, Changsha 410073, China

Abstract. This paper first discusses the structure of abstract smoothing splines associated with bounded linear operators. The minimum-norm property and the representation of the operator smoothing spline are obtained by introducing a new inner product. Then the smoothing approximate solution with interpolating errors of operator equation $Tx=y$ is studied, and the error estimates are also given.

Keywords: Hilbert space, Smoothing spline, Operator Smoothing approximate solution.

(*Received August 05, 2008, accepted September 30, 2008*)

Suppose X ,Y are real Hilbert spaces, T is a bounded linear operator from X to Y. There is an important problem both in theory and application: for a given $y\in Y$, find a approximate solution for the operator equation $Tx = y$. Paper [1] summarized the general theory of the projection approximate solution for the operator equation. [2] established the theory and method of using the operator of abstract spline interpolations for the approximate solution of the operator equation and indicated that solution of $Tx = y$ can be approximated by operator interpolation splines $S_n x$ which satisfy the interpolation conditions $\lambda_i\left(S_n x\right)=\lambda_i x\left(1\le i\le n\right)$.sometimes we needn't the solution strictly satisfies the interpolation conditions, that is there can be interpolating errors, but we need the solution has other optimum properties (e.g. smoothing rule). How can we construct such solutions? How can we estimate their errors? These are what we will discuss in this paper.

1. Operator Smoothing Spline and Minimum-Norm Problem

Let X,Y be real Hilbert spaces with inner products and associated norms respectively $\langle\cdot,\cdot\rangle_X,\langle\cdot,\cdot\rangle_Y,\|\cdot\|_X,\|\cdot\|_Y$. Denote by $\|T\|_X$ the norm of the bounded linear operator $T: X\to Y$, and suppose its range $R(T)=Y$, the null space $N(T)$ is an m-dimensional subspace of X . Suppose $\lambda_j\left(j=1,2\cdots\right)$ are linearly independent functionals on X with $\left\|\lambda_j\right\|_X$ satisfying $\sum_{j=1}^{\infty}\left\|\lambda_j\right\|_x^2<\infty$, and $\lambda_j\left(1\le j\le m\right)$ are linearly independent on $N(T)$. For $x_0\in X$, and a positive integer $n\geqslant m$, if $s_n\in X$ solves the following optimization problem:

$$\min_{x\in X}\left\{\|Tx\|_Y^2+\sum_{j=1}^{n}(\lambda_j x_0-\lambda_j x)^2\right\}, \qquad (1)$$

we call $s_n\in X$ the operator T- smoothing spline[3] with respect to x_0 and $\{\lambda_j\}_1^n$.

For a given $y\in Y$, denote by x_0 the solution of operator equation $Tx= y$. Suppose we have the values $\lambda_j x_0\left(1\le j\le m\right)$, and $x_n\in X$ is the solution of optimization problem (1) satisfying $\left\|x_n-x_0\right\|_x\to 0\left(n\to\infty\right)$, we call x_n the **smoothing approximate solution** of $Tx=y$.

Introduce new spaces as

+ *E-mail address:* xjz_20075@163.com

$$X^+ = \{x^+ = (x,e) : x \in X, e = (e_1, e_2, \cdots, e_n, \cdots) \in l^2\}$$
$$Y^+ = \{y^+ = (y,p) : y \in Y, p = (p_1, p_2, \cdots, p_n, \cdots) \in l^2\}$$

Now, consider the operator $T^+ : X^+ \to Y^+$ and linear functionals λ_j^+ as follows

$$T^+x^+ = T^+(x,e) = (Tx, e - \tfrac{1}{2}\lambda x), \text{ where } \lambda x = (\lambda_1 x, \lambda_2 x, \cdots) \quad (2)$$

$$\lambda_j^+ x^+ = \lambda_j^+(x,e) = \tfrac{1}{2}\lambda_j x + e_j, \quad j = 1,2\cdots \quad (3)$$

It is clear that the null space $N(T^+) = \{x^+ = (x,e) : x \in N(T), e = \frac{1}{2}\lambda x\}$, and that $\lambda_j^+ \ (j = 1,2,\cdots)$ are linearly independent on X^+, $\lambda_j^+ (1 \le j \le m)$ are linearly independent on N（T）.

Let $\{\varphi_j\}_1^m$ be a basis for N（T）, then $\varphi_i^+ = (\varphi_i, \frac{1}{2}\lambda\varphi_i)\ (i = 1,2,\cdots,m)$ is a basis for N（T^+）, and we can know from the independence of the $\lambda_i (i = 1,2,\cdots,m)$ on N （T） that the matrix $\Phi = (\lambda_j^+ \varphi_i^+)_{m\times m} = (\lambda_j \varphi_i)_{m\times m}$ is invertible. Then denote by Φ_j the matrix which obtained by replacing the *j*-th column of Φ by $(\varphi_1, \cdots, \varphi_m)^T$, let $\alpha_j = \det \Phi_j / \det \Phi$, $\alpha_j^+ = (\alpha_j, \frac{1}{2}\lambda\alpha_j)\ (j = 1,2,\cdots,m)$, then

$$T^+ a_j^+ = 0,\ \lambda_i^+ \lambda_j^+ = \lambda_i a_j = \delta_{ij}, \quad 1 \le i, j \le m \quad (4)$$

Lemma 1. Any $x \in X^+$ and $x^+ = (x,e) \in X^+$ can be uniquely decomposed as

$$x = x_1 + x_2 = \sum_{i=1}^m (\lambda_i x) a_i + \bar{x}, \quad x_1 \in N(T), x_2 \in N(T)^+ \quad (5)$$

$$x^+ = \sum_{i=1}^m (\lambda_i x_1) a_i^+ + (x_2 e - \tfrac{1}{2}\lambda x_1) = \sum_{i=1}^m (\tfrac{1}{2}\lambda_i x + e_i) a_i^+ + \bar{x}^+ \quad (6)$$

and $\lambda_j \bar{x} = \lambda_j^+ \bar{x}^+ = 0 (1 \le j \le m), T\bar{x} = Tx, T^+\bar{x}^+ = T^+x^+$.

By Lemma 1, we define inner products for X, Y^+ and X^+ respectively as follows

$$\langle x, z\rangle_1 = \sum_{i=1}^m (\lambda_i x)(\lambda_i z) + \langle Tx, Tz\rangle_Y, \quad x, z \in X \quad (7)$$

$$\langle (y,p),(w,q)\rangle_* = \tfrac{1}{2}\langle y, w\rangle_Y + \sum_{j=1}^\infty p_j q_j, \quad (y,p),(w,q) \in Y^+ \quad (8)$$

$$\langle x^+, z^+\rangle = \sum_{i=}^m (\lambda_i^+ x^+)(\lambda_i^+ z^+) + \langle T^+ z^+\rangle_* = \sum_{i=1}^m (\tfrac{1}{2}\lambda_i x + e_i)(\tfrac{1}{2}\lambda_i z + d_i)$$
$$+ \tfrac{1}{2}\langle Tx, Tz\rangle_Y + \sum_{i=1}^\infty (e_i - \tfrac{1}{2}\lambda_i x)(d_i - \tfrac{1}{2}\lambda_i z), \quad x^+, z^+ \in X^+ \quad (9)$$

with the corresponding norms denoted by $\|\cdot\|_1, \|\cdot\|_*$ and $\|\cdot\|$ respectively. We know from the Lemma 1 and Lemma 2 in paper [2] that relative to the inner product (7) X is complete and the linear functionals $\lambda_i (j = 1,2,\cdots)$ are continuous. Furthermore, we can easily know that Y^+ is complete relative to the inner product (8).

Using the Banach's inverse mapping theorem, we can obtain the following lemma.

Lemma 2. The norms $\|\cdot\|_X, \|\cdot\|_1$ for X are equivalent.

We know from (2),(8) and (9) that the linear operator T^+: $X^+ \to Y^+$ is bounded, and $\|T^+\| \le 1$.

Now, we can obtain an important result as below. The proof is a generalization of that in [1] and is thus omitted.

Theorem 1. X^+ with the inner product (9) is a Hilbert space.

For $x_0 \in X$, denote

$$U_n^+(x_0) = \{x^+ = (x,e) \in X^+ : \lambda_j^+ x^+ = \lambda_j x_0 (1 \le j \le n), e_j = \tfrac{1}{2}\lambda_j x (j \ge n+1)\} \quad (13)$$

$$U_n^+(0) = \{x^+ = (x,e) \in X^+ : \lambda_j^+ x^+ = 0 (1 \le j \le n), e_j = \tfrac{1}{2}\lambda_j x (j \ge n+1)\} \quad (14)$$

then the optimization problem（1）can be described as the minimum-norm problem in space $(X^+, \|\cdot\|)$:

$$\min_{x\in X}\{\|Tx\|_Y^2 + \sum_{j=1}^{n}(\lambda_j x_0 - \lambda_j x)^2\} = \min_{(x,e)\in U_n^+(0)}\{\|Tx\|_Y^2 + \sum_{j=1}^{n}(e_j - \tfrac{1}{2}\lambda_j x)^2\}$$
$$= \min_{(x,e)\in U_n^+(0)}\{\|Tx\|_Y^2 + \sum_{j=1}^{\infty}(e_j - \tfrac{1}{2}\lambda_j x)^2 + \sum_{j=1}^{m}(\tfrac{1}{2}\lambda_j x + e_j)^2\}$$
$$= \min_{(x,e)\in U_n^+(0)}\|x\|^2 \qquad (15)$$

By introducing new spaces X^+, Y^+, the optimization problem（1）can be reformulated as a minimum-norm problem in space X^+. This method was inspired by Weinert and Sidhu [4],[5]. But their method is restricted to the ordinary differential operator spline, and n is fixedly, we explore abstract spaces and abstract operators. In order to get the smoothing spline sequence and the smoothing approximate solution sequence to the operator equation, associated norms and spline projection methods have to be adapted to the arbitrariness of n. All above lead to the fact that our work (section1 and 2) is essentially different to the work in papers [4][5].

2. Solve Operator Smoothing Spline

Lemma 3. Assume that h_j is the representer in $(X, \langle\cdot\rangle_1)$ of the functional λ_j, let

$$h_j^+ = (g_j, w_j) = \begin{cases}(h_j, \tfrac{1}{2}\lambda h_j), & 1\le j\le m\\ (2h_j, w_j), & j\ge m+1\end{cases} \qquad (16)$$

where $w_j = (0,\cdots,0,\lambda_{m+1}h_j,\cdots,\lambda_{j-1}h_j, 1+\lambda_j h_j, \lambda_{j+1}h_j,\cdots)\ (j\ge m+1)$, then

$$\lambda_j^+ x^+ = \langle \lambda_j^+, x^+\rangle, \quad x^+\in X^+,\ j=1,2,\cdots$$

Lemma 3 shows that h_i^+ is the representer of λ_i^+, so λ_i^+ is continuous with respect to the inner (9) in space X^+.

Theorem 2. If $x^+ = (x,e)\in U_n^+(0)^\perp$, and $e_j = \frac{1}{2}\lambda_j x(j\ge n+1, n\ge m)$, then x^+ can be represented linearly by $h_1^+, h_2^+,\cdots,h_n^+$.

Proof : For any $z^+ = (z,d)\in X^+$ and every $n\ge m+1$, we have from (16) that

$$\langle h_i^+, z^+\rangle = \lambda_i^+ z^+ + \sum_{j=n+1}^{\infty}(w_{ij} - \lambda_j h_i)(d_j - \tfrac{1}{2}\lambda_j z), \quad m+1\le i\le n \qquad (17)$$

when $z^+\in U_n^+(0)$, since $d_j = \frac{1}{2}\lambda_j z\ (j\ge n+1), \lambda_i^+ z^+ = 0\ (1\le i\le n)$, then for any $w_{ij} = (j\ge n+1)$, we have $\sum_{j=n+1}^{\infty}(w_{ij} - \lambda_j h)(d_j - \frac{1}{2}\lambda_j z) = 0$, and $\langle h_i^+, z^+\rangle = \lambda_i^+ z^+ = 0\ (m+1\le i\le n)$. In a similar way, we can know that $\langle h_i^+, z^+\rangle = 0 (1\le i\le m)$ hold for any $z^+\in U_n^+(0)$ and $w_{ij} = (j\ge n+1)$. Then let

$$\overline{h}_i^+ = (h_i, \overline{w}_i)\quad (1\le i\le m), \qquad \overline{h}_i^+ = (2h_i, \overline{w}_i)\quad (m+1\le i\le n) \qquad (18)$$
$$\overline{w}_i = (\tfrac{1}{2}\lambda_j h_i,\cdots,\tfrac{1}{2}\lambda_n h_i, \overline{w}_{i,n+1}, \overline{w}_{i,n+2},\cdots), \quad 1\le i\le m$$
$$\overline{w}_i = (0,\cdots,0,\lambda_{m+1}h_i,\cdots,\lambda_{i-1}h_i, 1+\lambda_i h_i, \lambda_{i+1}h_i,\cdots,\lambda_n h_i, \overline{w}_{i,n+1}, \overline{w}_{i,n+2},\cdots),\ (m+1\le i\le n)$$

where $\overline{w}_{ij}(1\le i\le n, j\ge n+1)$ are arbitrary, thus, when $z^+\in U_n^+(0)$, we have

$$\langle \overline{h}_i^+, z^+\rangle = 0\ \ (1\le i\le n)\ .$$

On the contrary, if $z^+ = (z,d)$ satisfies $\langle \overline{h}_i^+, z^+\rangle = 0 (1\le i\le n)$, then there must be $\langle \overline{h}_i^+, z^+\rangle = 0 (1\le i\le n)$, which implies, using Lemma 3, $\lambda_i^+ z^+ = 0 (1\le i\le n)$. And similar to (17), we

know that $0=\langle \bar{h}_i^+, z^+\rangle = \sum_{j=n+1}^{\infty}(\bar{w}_{ij}-\lambda_j h)(d_j-\frac{1}{2}\lambda_j z)=0$ holds for any $\bar{w}_{ij}(j\ge n+1)$. Setting $\bar{w}_{ij}=\lambda_j h_i + d_j - \frac{1}{2}\lambda_j z (j\ge n+1)$, gives $\sum_{j=n+1}^{\infty}(d_j-\frac{1}{2}\lambda_j z)^2=0$,
then $d_j=\frac{1}{2}\lambda_j z(j\ge n+1)$, so $z^+\in U_n^+(0)$.

From above we have $z^+\in U_n^+(0)\Leftrightarrow \langle \bar{h}_i^+, z^+\rangle=0(1\le i\le n)$. It follows that if we denote $\bar{H}_n^+=Span\{\bar{h}_i^+, 1\le i\le n\}$, then

$$U_n^+(0)=(\bar{H}_n^+)^{\perp}. \tag{19}$$

We will prove that $\bar{H}_n^+$ is a closed set. Let $\bar{g}_i^+=\bar{h}_i^+-h_i^+$, that is $\bar{h}_i^+=h_i^++\bar{g}_i^+$, where h_i^+ is defined as (16). Using (9), (16) and (18), we have $\|\bar{h}_i^+\|^2=\|h_i^+\|^2+\|\bar{g}_i^+\|^2$. Denote

$$H_n^+=Span\{\bar{h}_i^+, 1\le i\le n\},\quad E_n^+=Span\{\bar{g}_i^+, 1\le i\le n\}$$

thus $\bar{H}_n^+$ can be expressed as the direct sum : $\bar{H}_n^+=H_n^+\oplus E_n^+$. Noticing that $\bar{g}_i^+=(0,\bar{\theta}_i)$,

$\bar{\theta}_i=(0,\cdots,0,\ \bar{w}_{i,n+1}-\lambda_{n+1}g_i, \bar{w}_{i,n+2}-\lambda_{n+2}g_i,\cdots)$, $\|\bar{g}_i^+\|^2=\sum_{j=n+1}^{\infty}(\bar{w}_{ij}-\lambda_j g_i)^2$, and the arbitrariness of $\bar{w}_{ij}$, we can regard E_n^+ as the space l^2 , so E_n^+ is also a complete space. Since H_n^+ is a finite dimensional subspace, we know that $\bar{H}_n^+$ is a closed set of X^+.

Now, it can be know from (19) that $U_n^+(0)^{\perp}=(\bar{H}_n^+)^{\perp\perp}=\bar{H}_n^+$, which implies that when $x^+=(x,e)\in U_n^+(0)^{\perp}$, x^+ must be a linear combination of $\bar{h}_1^+,\cdots,\bar{h}_n^+$. We know from (16),(18) that when $1\le i\le n$, then $h_1^+=(g_1,w_i)\in U_n^+(0)^{\perp}$ and $w_{ij}=\frac{1}{2}\lambda_j g_i(j\ge n+1)$ .Consequently, if $x^+=(x,e)\in U_n^+(0)^{\perp}$ satisfying $e_j=\frac{1}{2}\lambda_j x(j\ge n+1)$, then x^+ can be expressed by a linear combination of $\bar{h}_1^+,\cdots,\bar{h}_n^+$. ∎

Theorem 3. The operator T -smoothing spline $s_n^+=(s_n,e_n^*)$ defined by (1) or (15) is the projection of any element in $U_n^+(x_0)$ onto $U_n^+(0)^{\perp}$, and can be expressed as a linear combination of $\bar{h}_1^+,\cdots,\bar{h}_n^+$, i.e.

$$s_n^+=(s_n,e_n^*)=\sum_{i=1}^{n}c_i h_i^+ \tag{20}$$

Proof : For any $\tilde{x}^+\in U_n^+(x_o)$, since $U_n^+(0)$ is a closed subspace, by the projection theorem, there exists a unique $\hat{x}^+\in U_n^+(0)$ satisfies

$$\|\tilde{x}^+-\hat{x}^+\|=\inf_{z^+\in U_n^+(0)}\|\tilde{x}^+-z^+\|$$

where $\tilde{x}^+-\hat{x}^+\in U_n^+(0)^{\perp}$. It is clear that $\tilde{x}^+-\hat{x}^+$ and $\tilde{x}^+-z^+$ belong to $U_n^+(x_0)$, and $U_n^+(x_0)=\tilde{x}^+-U_n^+(0)$, these imply, from (15), that $\tilde{x}^+-\hat{x}^+$ must be the operator *T*- smoothing spline $s_n^+=(s_n,e_n^*)$ with $e_{nj}^*=\frac{1}{2}\lambda_j s_n(j\ge n+1)$. Now, by Theorem 2, $s_n^+=(s_n,e_n^*)$ can be expressed by a linear combination of $h_1^+,\cdots,h_n^+$. ∎

From (20) and (4), we know

$$e_n^*=\frac{1}{2}\sum_{i=1}^{m}c_i(\lambda h_i)+\sum_{i=m+1}^{n}c_i w_i, s_n=\sum_{i=1}^{m}c_i h_i+2\sum_{i=m+1}^{n}c_i h_i ,$$

If e_n^* is denoted as $e_n^*=(e_{n1}^*,e_{n2}^*,\cdots,e_{nj}^*,\cdots)$, then from (4) we have

$$e_{nj}^{*}=\tfrac{1}{2}\sum_{i=1}^{m}c_i(\lambda_j h_i)=\tfrac{1}{2}c_j,\qquad 1\le j\le m$$
$$e_{nj}^{*}=c_j+\tfrac{1}{2}\sum_{i=1}^{n}c_i(\lambda_j g_i)=c_j+\tfrac{1}{2}\sum_{i=1}^{m}c_i(\lambda_j h_i)+\sum_{i=m+1}^{m}c_i(\lambda_j h_i),m+1\le j\le n \qquad (21)$$
$$e_{nj}^{*}=\tfrac{1}{2}\sum_{i=1}^{n}c_i(\lambda_j g_i)=\tfrac{1}{2}\sum_{i=1}^{m}c_i(\lambda_j h_i)+\sum_{i=m+1}^{m}c_i(\lambda_j h_i),\ \ j\ge n+1$$

Since $s_n^+=(s_n,e_n^*)\in U_n^+(x_0)$, then $e_{nj}^*=\lambda_j x_0-\frac{1}{2}\lambda_j s_n(1\le j\le n)$, and by (20)(21), we have

$$c_j+\sum_{i=m+1}^{n}c_i(\lambda_j h_i)=\lambda_j x_0,\ 1\le j\le m$$
$$c_j+\sum_{i=m+1}^{n}c_i(\lambda_j h_i)=\lambda_j x_0-\sum_{i=1}^{m}c_i(\lambda_j h_i),\ m+1\le j\le n \qquad (22)$$

Substituting $c_i(1\le i\le n)$ determined by (22) into (20), we obtain the operator *T*-smoothing spline s_n^+. It follows from $\frac{1}{2}\lambda_j s_n+e_{nj}^*=\lambda_j x_0$, (21) and (22) that the interpolation errors of s_n and x_0 with respect to $\lambda_j(1\le j\le n)$ are

$$\lambda_j x_0-\lambda_j s_n=c_j-\lambda_j x_0=-\sum_{i=m+1}^{n}c_i(\lambda_j h_i),\ 1\le j\le m$$
$$\lambda_j x_0-\lambda_j s_n=c_j,\ m+1\ge j\ge n \qquad (23)$$

3. Smoothing Approximate Solution and Error Estimation for Operator Equation

Let $\{\tilde{h}_j^+\}$ be the orthonormal system in X^+ obtained by orthogonalizing $\{h_j^+\}$ with respect to inner product (9), we have

$$\tilde{h}_j^+=h_j^+=(h_j,\tfrac{1}{2}\lambda h_j)(1\le j\le m),\ \tilde{h}_j^+=\sum_{i=1}^{j}\beta_{ij}h_j^+,\beta_{jj}\ne 0(j\ge m+1) \qquad (24)$$

where β_{ij} are constants.

Denote $X_n=Span\{h_1,\cdots,h_n\}\subset(X,\langle\cdot\cdot\rangle_1)$, by Hahn-Banach theorem, there exist $l_i\in X_n$,

$$\langle l_i,h_j\rangle_1=\lambda_j l_i=\delta_{ij},i,j=1,2,\cdots \qquad (25)$$

let $l_i^+=(l_i,\frac{1}{2}\lambda l_i)\ (i=1,2.\cdots)$, then

$$\langle l_i^+,h_j^+\rangle=\lambda_j^+ l_i^+ +\tfrac{1}{2}\lambda_j l_i=\lambda_j l_i=\delta_{ij},\quad i,j=1,2,\cdots \qquad (26)$$

which implies, for $i,j\ge m+1$,

$$\langle T^+l_i^+,T^+h_j^+\rangle_*=\langle l_i^+,h_j^+\rangle=\delta_{ij} \qquad (27)$$

Denote $X_n^+=Span\{h_1^+,\cdots,h_n^+\}(n=1,2,\cdots)$, $Y_n^+=Span\{T^+h_{m+1}^+,\cdots,T^+h_n^+\}(n\ge m+1)$

Let $P_n:X^+\to X_n^+,Q_n:Y^+\to Y_n^+$ are projection operators:

$$P_n x^+=\sum_{j=1}^{n}\langle l_k^+,x^+\rangle h_j^+,\quad x^+\in X^+ \qquad (28)$$

$$Q_n y^+=\sum_{j=m+1}^{n}\langle T^+l_j^+,y^+\rangle_* T^+h_j^+,\quad y^+\in Y^+ \qquad (29)$$

Now, proceeding in a fashion analogous to that used in [2], the following theorem is readily established.

Theorem 4. The operator sequences $\{P_n\},\{Q_n\}$ are uniformly bounded, namely there exists a constant $C>0$, such that $\|P_n\|\le C,\|Q_n\|\le C$, and

$$\lim_{n\to\infty}\|P_n x^+-x^+\|=0 \qquad (30)$$

Let T_n^+ is the restriction of T^+ to X_n^+, then T_n^+ is a bounded linear operator from X_n^+ to Y_n^+.

Lemma 4. Suppose $y^+\in Y^+$, and $x^+\in X^+$ is a solution for $T^+x^+=y^+$, then all solutions for

$$T_n^+ x_n^+ = Q_n y^+, \quad x_n^+ \in X^+, n \geq m+1 \tag{31}$$

are given by

$$x_n^+ = \bar{x}^+ + \sum_{k=m+1}^{n} \langle l_k^+, x^+ \rangle h_k^+ = \bar{x}^+ + \sum_{k=m+1}^{n} \langle T^+ l_k^+, y^+ \rangle_* h_k^+ \tag{32}$$

where $\bar{x}^+$ is any element in $N(T^+)$.

Proof : Using (9), (27) gives

$$T_n^+ x_n^+ = \sum_{k=m+1}^{n} \langle l_k^+, x^+ \rangle T_n^+ h_k^+ = \sum_{k=m+1}^{n} \langle T^+ l_k^+, T^+ x^+ \rangle_* T^+ h_k^+ = Q_n y^+ .$$

Conversely, let $x_n^+ = \sum_{k=1}^{n} a_k h_k^+ \in X_n^+ (n \geq m+1)$ be a solution for the equation (31), denote $\tilde{X}_n^+ = Span\{h_{m+1}^+, \cdots h_n^+\}$, it is clear that $T_n^+ : \tilde{X}_n^+ \to Y_n^+$ is bijective. By Banach inverse operator theorem, we know T_n^+ has a bounded inverse $(T_n^+)^{-1}$. let $\bar{x}_n^+ = x_n^+ - \sum_{k=1}^{m} a_k h_k^+ \in \tilde{X}_n^+$, then we know from (31) that

$$\bar{x}_n^+ = (T_n^+)^{-1} Qny^+ = (T_n^+)^{-1} \sum_{j=m+1}^{n} \langle T^+ l_j^+, y^+ \rangle_* T^+ h_j^+ = \sum_{j=m+1}^{n} \langle l_j^+, x^+ \rangle h_j^+$$

and that

$$x_n^+ = \sum_{k=1}^{m} a_k h_k^+ + \sum_{j=m+1}^{n} \langle l_j^+, x^+ \rangle h_j^+ . \quad \blacksquare$$

Lemma 5. For every $y^+ \in Y^+$, denote $\tilde{x}_n^+ = \sum_{k=m+1}^{n} \left\langle T^+ l_k^+, y^+ \right\rangle_* h_k^+$, then equation $T^+ x^+ = y^+$ has a solution $\tilde{x}^+$ satisfying $\left\| \bar{x}_n^+ - \tilde{x}^+ \right\| \to 0 (n \to \infty)$, and all solutions of the equation are $\tilde{x}^+ + x_0^+$ $(\forall x_0^+ \in N(T^+))$.

Proof : Recalling that $R(T) = Y$, we know $R(T^+) = Y^+$, then for every $y^+ \in Y^+$, there exists $x^+ \in X^+$ such that $T^+ x^+ = y^+$. Let

$$\tilde{x}^+ = x^+ - \sum_{k=1}^{m} \langle l_k^+, x^+ \rangle h_k^+ , \tag{33}$$

then $T^+ \tilde{x}^+ = T^+ x^+ = y^+$. From (9), we have $\left\langle T^+ l_k^+, y^+ \right\rangle_* = \left\langle l_k^+, x^+ \right\rangle (k \geq m+1)$, thus $\tilde{x}_n^+ - \tilde{x}^+ = \sum_{k=1}^{n} \langle l_k^+, x^+ \rangle h_k^+ - x^+$. Furthermore we know from (28) and (30) that $\left\| \tilde{x}_n^+ - \tilde{x}^+ \right\| \to 0 (n \to \infty)$.
$\blacksquare$

Theorem 5. For every $y^+ = (y, d) \in Y^+$ and any given real numbers $r_k (1 \leq k \leq m)$, the unique solution of $T^+ x^+ = y^+$ satisfying $\lambda_k^+ x^+ = r_k (1 \leq k \leq m)$ is

$$x^+ = \sum_{j=m+1}^{\infty} \left[\sum_{i=1}^{m} \beta_{ij} r_i + \sum_{i=m+1}^{j} \beta_{ij} \left(\sum_{k=1}^{m} (\lambda_k h_i)(r_k - d_k) + (Th_i, y)_Y + d_i \right) \right] \sum_{k=1}^{j} \beta_{kj} h_k^+ + \sum_{k=1}^{j} \beta_{kj} h_k^+ + \sum_{k=1}^{m} r_k h_k^+$$

Proof : By taking $h_j^+ = \tilde{h}_j^+, l_i^+ = \tilde{h}_i^+$ in (26), we know that Lemma 4 and Theorem 5 hold for all l_i^+ and h_i^+ replaced by $\tilde{h}_i^+$. By Theorem 5, suppose the solution of $T^+ x^+ = y^+$ is

$$x^+ = \tilde{x}^+ + x_0^+, \qquad x_0^+ = \sum_{k=1}^{m} c_k h_k^+$$

where $\tilde{x}^+$ satisfies $\left\| \tilde{x}_n^+ - \tilde{x}^+ \right\| \to 0 (n \to \infty)$, and $\langle l_k^+, \tilde{x}_n^+ \rangle = \lambda_k^+ \tilde{x}^+ = 0 (1 \leq k \leq m)$. From the continuity of λ_k^+ we know $\lambda_k^+ \tilde{x}^+ = 0 (1 \leq k \leq m)$, so the solution of $T^+ x^+ = y^+$ which satisfies $\lambda_k^+ \tilde{x}^+ = r_k (1 \leq k \leq m)$ is unique，and it is given by

$$\tilde{x}^+ + \sum_{k=1}^{m} (\lambda_k^+ \tilde{x}^+) h_k^+ = \tilde{x}^+ + \sum_{k=1}^{m} r_k h_k^+ \tag{34}$$

By (33) and the assumption that $\tilde{h}_j^+ (j = 1, 2, \cdots)$ is a complete system, we can get

$$\tilde{x}^+ = \sum_{j=m+1}^{\infty}\langle \tilde{h}_j^+, x^+\rangle \tilde{h}_j^+ = \sum_{j=m+1}^{\infty}(\sum_{i=1}^{j}\beta_{ij}\langle h_i^+, x^+\rangle)\tilde{h}_j^+$$
$$= \sum_{j=m+1}^{\infty}[\sum_{i=1}^{m}\beta_{ij}r_i + \sum_{i=m+1}^{j}\beta_{ij}\langle h_i^+, x^+\rangle]\tilde{h}_j^+ \quad 35)$$

Then by $T^+x^+ = (Tx, e-\frac{1}{2}\lambda x) = (y,d)$, we have $Tx = y, e = \frac{1}{2}\lambda x + \mathrm{d}$, and it follows from (3),(9) and (16) that , for $i \ge m+1$, we have

$$\langle h_i^+, x^+\rangle = \sum_{j=1}^{m}(\lambda_j h_i)r_j + \tfrac{1}{2}\langle T(2h_i), Tx\rangle_Y$$
$$+\sum_{j=1}^{m}(-\tfrac{1}{2}\lambda_j(2h_i))(\lambda_j x + d_j - \tfrac{1}{2}\lambda_j x) + (\tfrac{1}{2}\lambda_i x + d_i - \tfrac{1}{2}\lambda_i x)$$
$$= \sum_{j=1}^{m}(\lambda_j h_i)(r_j - d_j) + \langle Th_i, y\rangle_Y + d_i$$

Substituting it into (34) and (35), we complete our proof. ∎

By the similar kind of argument used in [2], we have the following two lemmas.

Lemma 6. There exists a constant $q > 0$, such that for any $x_n^+ \in X_n^+$, if $x^+ \in X^+$ satisfying $P_n x^+ = x_n^+$, then $\|x_n^+\| \le q\|T^+x^+\|$, where $\tilde{x}_n^+ = \sum_{k=m+1}^{n}\langle l_k^+, x^+\rangle h_k^+$.

Lemma 7. For any $x \in X$, let $\hat{x}^+ = (x, \frac{1}{2}\lambda x) \in X^+$, then the smoothing spline satisfying $\lambda_i^+ s_n^+ = \lambda_i x (1 \le i \le n)$ is given by

$$s_n^+ = (s_n, e_n^*) = \sum_{j=1}^{n}\langle \tilde{h}_j^+, \hat{x}^+\rangle \tilde{h}_j^+ \quad (36)$$

Proof : From (20), we know $s_n^+ = \sum_{j=1}^{n}\tilde{c}_j\tilde{h}_j^+ = \sum_{j=1}^{n}\langle \tilde{h}_j^+, s_n^+\rangle \tilde{h}_j^+$. Thus, if $\lambda_i^+ s_n^+ = \lambda_i x$ $(1 \le i \le n)$, we have

$$\langle \tilde{h}_j^+, s_n^+\rangle = \sum_{i=1}^{j}\beta_{ij}(\lambda_i^+ s_n^+) = \sum_{i=1}^{j}\beta_{ij}(\lambda_j^+, \hat{x}^+) = (\tilde{h}_j^+, \hat{x}^+) \quad (37)$$

so (36) is true. ∎

According to (36) we can defined a mapping $\hat{S}_n : X \to X^+$, that is $\hat{S}_n(x) = s_n^+$. For any $x^+ = (x,e) \in X^+$, since $x^+ = \hat{x}^+ + (0, e - \frac{1}{2}\lambda x)$, then we have from (3),(24) and Lemma 3 that

$$\sum_{j=1}^{n}\left\langle \tilde{h}_j^+, x^+\right\rangle \tilde{h}_j^+ = \sum_{j=1}^{n}\left\langle \tilde{h}_j^+, \hat{x}^+\right\rangle \tilde{h}_j^+ + \sum_{i=1}^{m}\left(e_j - \tfrac{1}{2}\lambda_i x\right)h_j^+ + \sum_{j=m+1}^{n}\sum_{i=1}^{j}\beta_{ij}\left(e_i - \tfrac{1}{2}\lambda_i x\right)\tilde{h}_j^+$$
$$= \hat{S}_n x + \sum_{j=m+1}^{n}\sum_{i=1}^{j}\beta_{ij}(e_i - \tfrac{1}{2}\lambda_i x)\tilde{h}_j^+ + \sum_{j=1}^{m}(e_j - \tfrac{1}{2}\lambda_j x)h_j^+$$
$$= \tilde{S}_n x + \sum_{j=1}^{m}(e_j - \tfrac{1}{2}\lambda_j x)h_j^+ \quad (38)$$

where $\tilde{S}_n x = \hat{S}_n x + \sum_{j=m+1}^{n}\sum_{i=1}^{j}\beta_{ij}(e_i - \frac{1}{2}\lambda_i x)\tilde{h}_j^+$.

Theorem 6. For any $x^+ = (x,e) \in X^+$, we have

$$\left\|x^+ - \tilde{S}_n x - \sum_{j=1}^{m}(e_j - \tfrac{1}{2}\lambda_j x)h_j^+\right\| \le \left\|x^+ - P_n x^+\right\| \le (1+q)\left\|x^+ - \tilde{S}_n x - \sum_{j=1}^{m}(\lambda_j x - \left\langle l_j^+, x^+\right\rangle)h_j^+\right\|$$

Proof : It is clear from (38) that $\tilde{S}_n x + \sum_{j=1}^{m}(e_j - \frac{1}{2}\lambda_j x)h_j^+$ is the orthogonal projection of x^+ onto X_n^+. Since $P_n x^+ \in X_n^+$, so $\left\|x^+ - \tilde{S}_n x - \sum_{j=1}^{m}(e_j - \frac{1}{2}\lambda_j x)h_j^+\right\| \le \left\|x^+ - P_n x^+\right\|$.

For any $z_n^+ \in X_n^+$, we have $z_n^+ = P_n z_n^+$. Thus using Lemma 6 and $\|T^+\| \le 1$, we get

$$\left\|x^+ - P_n x^+\right\| = x^+ - \sum_{j=1}^{m}\left\langle l_j^+, x^+\right\rangle h_j^+$$
$$\le \left\|\tilde{x}_n^+ - \tilde{z}_n^+\right\| + \left\|x^+ - z_n^+ - \sum_{j=1}^{m}\left\langle l_j^+, x\right\rangle h_j^+\right\|$$

$$=q\left\|T^+\left(x^+-\tilde{z}_n^+-\sum_{j=1}^m\langle l_j^+,x\rangle h_j^+\right)\right\|+\left\|x^+-z_n^+-\sum_{j=1}^m\langle l_j^+,x\rangle h_j^+\right\|$$
$$\le (q+1)\left\|x^+-\sum_{j=1}^m\langle l_j^+,x\rangle h_j^+-\tilde{z}_n^+\right\|$$

which implies

$$\left\|x^+-p_nx^+\right\|\le(1+q)\inf_{z_n^+\in X_n^+}\left\|x^+-\sum_{j=1}^m\langle l_j^+,x\rangle h_j^+-\tilde{z}_n^+\right\|.$$

Therefore when $\tilde{z}_n^+$ is the orthogonal projection of $x^+-\sum_{j=1}^m\langle l_j^+,x^+\rangle h_j^+$ onto $\tilde{X}_n^+$ （denote by $\hat{x}^+$）, the infimum of the right side of above inequality is attained. It is clear that $\hat{z}_n^+=\sum_{j=n+1}^n\left\langle \tilde{h}_j,x^+\right\rangle\tilde{h}_j^+$, and by (38), we have

$$\left\|x^+-p_nx^+\right\|\le(1+q)\left\|x^+-\sum_{j=1}^m\langle l_j^+,x^+\rangle h_j^+-\sum_{j=m+1}^n\langle\tilde{h}^+,x^+\rangle\tilde{h}_j^+\right\|$$
$$=(1+q)\left\|x^+-\hat{S}_nx-\sum_{j=m+1}^n\sum_{i=l}^j\beta_{ij}\left(e_i-\tfrac{1}{2}\lambda_ix\right)\tilde{h}_j^++\sum_{j=1}^m(\lambda_ix)h_j^+-\sum_{j=1}^m\langle l_j^+,x^+\rangle h_j^+\right\|$$
$$=(1+q)\left\|x^+-\tilde{S}_nx+\sum_{j=1}^m\left(\lambda_jx-\langle l_j^+,x^+\rangle\right)h_j^+\right\|.$$

Theorem 7. Assume that x_0 is the solution of the equation $Ts=y$, s_n is the operator T-smoothing spline determined by (1) with respect to the x_0, then $\|s_n-x_0\|\to 0\ (n\to\infty)$. In other words, s_n is the smoothing approximate solution of $Tx=y$.

Proof: From (36), we know that the T-smoothing spline $s_n^+=(s_n,e_n^*)$determined by (1) or (15) is the projection onto X_n^+ of $\hat{x}_0^+=\left(x_0,\frac{1}{2}\lambda x_0\right)$ and $\left\|s_n^+-\hat{x}_0^+\right\|\to 0(n\to\infty)$. It is clear that s_n is just the T-smoothing spline with respect to x_0, and $s_n^+\in U_0^+(x_0)$, thus

$$\lambda_j^+s_n^+=\tfrac{1}{2}\lambda_js_n+e_{nj}^*=\lambda_jx_0\,(1\le j\le n),\quad e_{nj}^*=\tfrac{1}{2}\lambda_js_n\,(j\ge n+1).$$

It follows from $\left\|s_n^+-\hat{x}_0^+\right\|\to 0(n\to\infty)$ that

$$\left\|s_n^+-\hat{x}_0^+\right\|^2=\sum_{i=1}^m[\tfrac{1}{2}\lambda_i(s_n-x_0)+(e_{ni}^*-\tfrac{1}{2}\lambda_ix_0)]^2+\tfrac{1}{2}\left\|T(s_n-x_0)\right\|_Y^2$$
$$+\sum_{i=1}^\infty\left[(e_{ni}^*-\tfrac{1}{2}\lambda_ix_0)-\tfrac{1}{2}\lambda_i(s_n-x_0)\right]^2$$
$$=\tfrac{1}{2}\left\|Ts_n-Tx_0\right\|_Y^2+\sum_{i=1}^n(\lambda_ix_0-\lambda_is_n)^2\to 0,\ n\to\infty.$$

From (7), we have

$$\left\|s_n-x_0\right\|_1^2=\left\|Ts_n-Tx_0\right\|_y^2+\sum_{i=1}^m\left(\lambda_ix_0\right)^2\to 0,\quad n\to\infty$$

In light of Lemma 2, we have $\|s_n-x_0\|_x\to 0(n\to\infty)$. ■

We can compute $s_n^+=(s_n,e_n^*)$ by (20)—(22), and the interpolating errors $\lambda_jx_0-\lambda_js_n$ by (23). More exactly, the c_i in (20)—(23) should be denoted by c_{ni} $(1\le i\le n)$. As in [6], we can also establish a recursive algorithm for s_n^+.

If we let $\hat{x}^+=(x,\frac{1}{2}\lambda x)$, $\hat{y}^+=(y,0)$, then x is a solution of $Ts=y$ if and only if $\hat{x}^+$ is a solution of $T^+\hat{x}^+=\hat{y}^+$. It is clear that the results for $T^+x^+=y^+$ are all still hold for $T^+\hat{x}^+=\hat{y}^+$, and in this case the results are more simple. For example, corresponding to Theorem 6, for $T^+\hat{x}^+=\hat{y}^+$, we can know that the solution $\hat{x}^+=(x,\frac{1}{2}\lambda x)$, the best smoothing approximate solution $s_n^+=\hat{S}_nx$ and the projection solution

$P_n \hat{x}^+$ satisfy

$$\left\|\hat{x}^+ - \hat{S}_n x\right\| \le \left\|\hat{x}^+ - P_n \hat{x}^+\right\| \le (1+q)\left\|\hat{x}^+ - \hat{S}_n x - \tfrac{1}{2}\sum_{j=1}^{m}\left\langle Tl_j, y\right\rangle_Y h_j^+\right\|$$

where $y = Tx$.

4. References

[1] Mingjun Chen, Zhongxing Chen. *Operator Equations and Projective Approximate Solutions[M]*. Guangzhou: Guangdong Science Tech. Press, 1992.

[2] Xinjian Zhang. Operator Spline Interpolation and Approximate Solutions of Operator Equations on Hilbert Spaces[J]. *Acta Mathematica Sinica*. 2002, **45**(2): 227-234.

[3] Zhengxing Chen. *Data Fitting[M]*. Xian: Xian Jiaotong University Press. 1986.

[4] H. L.Wenert,R.H.Byrd and G.S.Sidhu. A Stochastic Framework for Recursive Computation of Spline Functions. II: Smoothing Splines[J]. *J. Optim. Theory Appl.*. 1980, **30**: 255-268

[5] G.S.Sidhu . Vector-Valjued Lg-Splines II: Smoothing Splines [J]. *J. of Mathematical Analysis and Applications*. 1994, **101**: 380-396

[6] Xinjian Zhang. On the Construction of Lg-Smoothing Splines[J]. *J. National university of defense Technology*. 1990, **12**(3): 32-38.