

# The Multiplicity of Eigenvalues of a Vectorial Sturm-liouville Problem<sup>\*</sup>

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**Abstract.** In this paper, we prove that under some conditions, A two-dimension vectorial Sturm-Liouville Problem can only have finitely many eigenvalues of multiplicity two. Using this result, we imply that the spectral of two Sturm-Liouville Problem of dimension one, or two string equation, have finitely many elements in common. And then we find a bound  $M_Q$  depending on  $Q(x)$ , such that the eigenvalues of the vectorial Sturm-Liouville Problem larger than  $M_Q$  are all simple.

**Keywords:** Vectorial Sturm-Liouville problems, Eigenvalues, Spectrum, Multiplicity, Potential function.

## 1. Introduction

In this paper, we study the following two-dimension vectorial Sturm-Liouville equation

$$Z''(x) + (\lambda I - Q(x))Z(x) = 0, \quad x \in [0, 1] \quad (1.1)$$

with boundary conditions

$$Z'(0) = Z'(1) = 0 \quad (1.2)$$

where  $I$  is the two-by-two identity matrix,  $Z(x)$  is two-dimension vectorial function,  $Q(x) = \begin{pmatrix} p_1(x) & -r(x) \\ -r(x) & p_2(x) \end{pmatrix}$  is a continuous two-by-two symmetric matrix-valued function define on  $[0, 1]$ , that is  $p_1(x), p_2(x), r(x)$  are continuous functions on  $[0, 1]$ .

It was well known that the spectrum of the following one-dimension Sturm-Liouville problem (1.3)(1.4) are all eigenvalues ([1][2][3][4][5]), and these eigenvalues are all simple.

$$y''(x) + (\lambda - q(x))y(x) = 0, \quad x \in [0, 1] \quad (1.3)$$

$$y'(0) = y'(1) = 0 \quad (1.4)$$

Thus two-dimension vectorial Sturm-Liouville problem (1.1)(1.2) may have the eigenvalues of multiplicity two. The eigenvalues of two-dimension vectorial Sturm-Liouville problem (1.1)(1.2) can be indexed as follows ([1][6][7])

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \quad (1.5)$$

But the numbers of eigenvalues of multiplicity two of (1.1)(1.2) are not well understood. Our purpose of this paper is to investigate the numbers of eigenvalues of multiplicity two of (1.1)(1.2) under some conditions.

The organization of this paper is as follows. Following this Introduction, we give in section 2 the numbers of eigenvalues of multiplicity two in Theorem 2.1:

Suppose  $Q(x) = \begin{pmatrix} p_1(x) & -r(x) \\ -r(x) & p_2(x) \end{pmatrix}$  in (1.1), if  $\int_0^1 r(x) dx \neq 0$ , then two-dimension vectorial Sturm-Liouville

Problem (1.1)(1.2) can only have finitely many eigenvalues of multiplicity two. And then we find a

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bound  $M_Q$  depending on  $Q(x)$ , such that the eigenvalues of the two-dimension vectorial Sturm-Liouville Problem larger than  $M_Q$  are all simple in Theorem 2.2: Suppose

$$Q_* = \int_0^1 \|Q(t)\| dt, R_* = \frac{|r(1)|}{2} + \frac{\int_0^1 |r'(x)| dx}{2} + 16e^2 Q_* \int_0^1 |r(x)| dx, M_Q = \max\{2Q_*, \frac{R_*}{|\int_0^1 r(x) dx|}\},$$

if  $\lambda_n$  is the eigenvalue of (1.1)(1.2) which satisfies the condition  $\sqrt{\lambda_n} > M_Q$ , then  $\lambda_n$  is simple.

Finally, in section 3, we apply Theorem 2.1 to study the intersection of two potential equation, or to two string equation. We obtain Theorem 3.1 as follows: Let  $\sigma(q)$  be the spectrum of Sturm-Liouville problem (1.3)(1.4). If  $\int_0^1 q_1(x) dx \neq \int_0^1 q_2(x) dx$ , then  $\sigma(q_1)$  and  $\sigma(q_2)$  only have finitely many elements in common.

And Theorem 3.2 : Suppose  $\rho_1(x)$  and  $\rho_2(x)$  are two real-value continuous functions on  $[0,1]$ , if  $\rho_1(x)$  and  $\rho_2(x)$  satisfy the following two conditions,  $\int_0^1 \rho_1^{1/2}(x) dx = \int_0^1 \rho_2^{1/2}(x) dx$ ,  $\int_0^1 \rho_1^{-10/4}(x) dx \neq \int_0^1 \rho_2^{-10/4}(x) dx$ , then  $\sigma(\rho_1)$  and  $\sigma(\rho_2)$  only have finitely many elements in common.

## 2. Numbers of eigenvalues with Multiplicity two

To study eigenvalue problems of (1.1)(1.2), we first consider the following Matrix differential equation ([1])

$$Y''(x) + (\lambda I - Q(x))Y(x) = 0, x \in [0,1] \quad (2.1)$$

with initial conditions

$$Y(0) = I, Y'(0) = 0 \quad (2.2)$$

where  $Y(x)$  is a two-by-two matrix-valued function.

Let  $Y(x, \lambda) = \begin{pmatrix} y_{11}(x, \lambda) & y_{12}(x, \lambda) \\ y_{21}(x, \lambda) & y_{22}(x, \lambda) \end{pmatrix}$  be the solution of the initial valued problem (2.1)(2.2), then  $y_1(x, \lambda) = \begin{pmatrix} y_{11}(x, \lambda) \\ y_{21}(x, \lambda) \end{pmatrix}$ ,  $y_2(x, \lambda) = \begin{pmatrix} y_{12}(x, \lambda) \\ y_{22}(x, \lambda) \end{pmatrix}$  are solutions of two-dimension vectorial Sturm-Liouville Problem (1.1) and satisfy  $y_1'(0, \lambda) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,  $y_2'(0, \lambda) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ . If  $\lambda_*$  is a simple eigenvalue of (1.1)(1.2) and  $y_1(x, \lambda)$  is the corresponding eigenfunction, then  $Y'(1, \lambda_*) = \begin{pmatrix} 0 & y'_{12}(x, \lambda) \\ 0 & y'_{22}(x, \lambda) \end{pmatrix}$ , and if  $\lambda_*$  is an eigenvalue of (1.1)(1.2) of multiplicity two, then  $Y'(1, \lambda_*)$  is the zero matrix. The following asymptotic formula of  $Y(x, \lambda)$  ([1][6]) will be used later: for  $\lambda > 0$ ,

$$Y(x, \lambda) = \begin{pmatrix} \cos \sqrt{\lambda} x + O(\frac{1}{\sqrt{\lambda}}) & O(\frac{1}{\sqrt{\lambda}}) \\ O(\frac{1}{\sqrt{\lambda}}) & \cos \sqrt{\lambda} x + O(\frac{1}{\sqrt{\lambda}}) \end{pmatrix} \quad (2.3)$$

The following Lemma shall be used to prove the main result of this paper.

### Lemma 2.1

If  $\lambda_*$  is an eigenvalue of (1.1)(1.2) of multiplicity two,  $y_1(x, \lambda) = \begin{pmatrix} y_{11}(x, \lambda) \\ y_{21}(x, \lambda) \end{pmatrix}$ ,  $y_2(x, \lambda) = \begin{pmatrix} y_{12}(x, \lambda) \\ y_{22}(x, \lambda) \end{pmatrix}$  are eigenfunctions corresponding to  $\lambda_*$ , then

$$\int_0^1 r(x)(y_{11}(x, \lambda)y_{22}(x, \lambda) - y_{12}(x, \lambda)y_{21}(x, \lambda)) dx = 0 \quad (2.4)$$

**Proof :** Because  $y_1(x, \lambda)$ ,  $y_2(x, \lambda)$  satisfy equation (1.1), we have

$$y''_{11}(x) + (\lambda_* - p_1(x))y_{11}(x) + r(x)y_{21}(x) = 0 \quad (2.5)$$

$$y''_{12}(x) + (\lambda_* - p_1(x))y_{12}(x) + r(x)y_{22}(x) = 0 \quad (2.6)$$

By (2.5) and (2.6) we have

$$y_{12}(x)y''_{11}(x) - y''_{12}(x)y_{11}(x) = r(x)(y_{11}(x)y_{22}(x) - y_{12}(x)y_{21}(x))$$

that is

$$\frac{d}{dx}(y_{12}(x)y'_{11}(x) - y'_{12}(x)y_{11}(x)) = r(x)(y_{11}(x)y_{22}(x) - y_{12}(x)y_{21}(x)) \quad (2.7)$$

Integrating (2.7) from  $x = 0$  to  $x = 1$  and using the boundary condition  $y'_{ij}(0) = y'_{ij}(1) = 0$ , we obtain (2.4).

### Lemma 2.2(Riemann-lebesgue Lemma [8])

If  $f$  be a Riemann-integrable function defined on interval  $[a, b]$ , then  $\lim_{|\alpha| \rightarrow \infty} \int_a^b f(x) \cos \alpha x dx = 0$  and  $\lim_{|\alpha| \rightarrow \infty} \int_a^b f(x) \sin \alpha x dx = 0$ , or to the complex form  $\lim_{|\alpha| \rightarrow \infty} \int_a^b f(x) e^{i\alpha x} dx = 0$ .

Now we give the main result of this paper.

### Theorem 2.1

If  $\int_0^1 r(x) dx \neq 0$ , then two-dimension vectorial Sturm-Liouville Problem (1.1)(1.2) can only have finitely many eigenvalues of multiplicity two.

**Proof :** On the contrary, suppose there are infinitely many eigenvalues  $\lambda_{n_k}$  which have multiplicity two. Denote the solution of (2.1) for  $\lambda = \lambda_{n_k}$  by  $Y(x, \lambda_{n_k}) = [y_{ij}(x, \lambda_{n_k})]_{i,j=1}^2$ . By (2.3) we have

$$\det Y(x, \lambda_{n_k}) = \frac{1 + \cos 2\sqrt{\lambda_{n_k}} x}{2} + O\left(\frac{1}{\sqrt{\lambda_{n_k}}}\right)$$

Because of Lemma 2.1 we obtain

$$\int_0^1 r(x) \left( \frac{1 + \cos 2\sqrt{\lambda_{n_k}} x}{2} + O\left(\frac{1}{\sqrt{\lambda_{n_k}}}\right) \right) dx = 0$$

then

$$\int_0^1 r(x) dx = - \left( \int_0^1 r(x) \cos 2\sqrt{\lambda_{n_k}} x dx + O\left(\frac{1}{\sqrt{\lambda_{n_k}}}\right) \right) \quad (2.8)$$

We let  $\lambda_{n_k} \rightarrow \infty$  in (2.8), and by Lemma 2.2( Riemann-Lebesgue Lemma([8])) we have

$$\int_0^1 r(x) dx = - \lim_{\lambda_{n_k} \rightarrow \infty} \int_0^1 r(x) \cos 2\sqrt{\lambda_{n_k}} x dx = 0 \quad (2.9)$$

This is a contradiction. Therefore, if  $\int_0^1 r(x) dx \neq 0$ , two-dimension vectorial Sturm-Liouville Problem (1.1)(1.2) can only have finitely many eigenvalues of multiplicity two.

The sequence of eigenvalues of (1.1)(1.2) can be indexed as follows

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$$

By Theorem 2.1, there exists a constant  $M_Q$  such that for all eigenvalues satisfying  $\lambda_n > M_Q$  are simple. Now we shall try to find a bound estimate of the constant  $M_Q$ .

Let  $A = (a_{ij})_{i,j=1}^2$  be a two-by-two matrix. Define the maximum norm of  $A$  as follows:

$$A = \sup\{|a_{ij}| : 1 \leq i, j \leq 2\}$$

If  $A$  and  $B$  are two two-by-two matrices, then we have

$$\|AB\| \leq 2 \|A\| \|B\| \quad (2.10)$$

Let  $Y(x, \lambda) = \begin{pmatrix} y_{11}(x, \lambda) & y_{12}(x, \lambda) \\ y_{21}(x, \lambda) & y_{22}(x, \lambda) \end{pmatrix}$  be the solution of the initial valued problem (2.1)(2.2), then  $Y(x, \lambda)$

satisfies the following integral equation ([7])

$$Y(x, \lambda) = \cos \sqrt{\lambda} x I + \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} Q(t) Y(t, \lambda) dt \quad (2.11)$$

Denote

$$Y(x, \lambda) = \cos \sqrt{\lambda} x I + G(x, \lambda) \quad (2.12)$$

where

$$G(x, \lambda) = [g_{ij}]_{ij=1}^2 = \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} Q(t) Y(t, \lambda) dt \quad (2.13)$$

The following Lemma shall be used to prove the main result of the paper.

### Lemma 2.3. (Gronwall's inequality ([9]))

Let  $I$  denote an interval of the real line of the form  $[a, \infty)$  or  $[a, b]$  or  $[a, b]$  with  $a < b$ . Let  $\alpha, \beta, u$  be real-valued functions defined on  $I$ . Assume that  $\beta, u$  are continuous and that the negative part of  $\alpha$  is integrable on every closed and bounded subinterval of  $I$ .

(a) If  $\beta$  is non-negative and if  $u$  satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_a^t \beta(s) u(s) ds, \quad t \in I$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s) \beta(s) \exp \int_s^t \beta(r) dr ds, \quad t \in I$$

(b) If in addition, the function  $\alpha$  is constant, then

$$u(t) \leq \alpha \exp \int_a^t \beta(s) ds, \quad t \in I$$

### Theorem 2.2

Suppose  $Q_* = \int_0^1 \|Q(t)\| dt$ ,  $R_* = \frac{|r(1)|}{2} + \frac{\int_0^1 |r'(x)| dx}{2} + 16e^2 Q_* \int_0^1 |r(x)| dx$ ,  $M_Q = \max\{2Q_*, \frac{R_*}{|\int_0^1 r(x) dx|}\}$

, if  $\lambda_n$  is the eigenvalue (1.1)(1.2) which satisfies the condition  $\sqrt{\lambda_n} > M_Q$ , then  $\lambda_n$  is simple.

**Proof :** The proof is by contradiction. Suppose  $\lambda_n$  is an eigenvalues of (1.1)(1.2) with multiplicity two and satisfies  $\sqrt{\lambda_n} > M_Q$ , i.e.,  $\sqrt{\lambda_n} > 2Q_*$ ,  $\sqrt{\lambda_n} > R_* / |\int_0^1 r(x) dx|$ . By Lemma 2.1 we have

$$\int_0^1 r(x) \det Y(x, \lambda_n) dx = 0 \quad (2.14)$$

Using (2.12) and (2.13),

$$\begin{aligned} \det Y(x, \lambda_n) &= \cos^2 \sqrt{\lambda_n} x + \cos \sqrt{\lambda_n} x \text{Trace} G(x, \lambda_n) + \det G(x, \lambda_n) \\ &= \frac{1 + \cos 2\sqrt{\lambda_n} x}{2} + h(x, \lambda_n) \end{aligned} \quad (2.15)$$

where  $h(x, \lambda_n) = \cos \sqrt{\lambda_n} x \text{Trace} G(x, \lambda_n) + \det G(x, \lambda_n)$ .

By (2.14) and (2.15) we have

$$\int_0^1 r(x) dx = - \left( \int_0^1 r(x) \cos 2\sqrt{\lambda_n} x dx + 2 \int_0^1 r(x) h(x, \lambda_n) dx \right) \quad (2.16)$$

then

$$\left| \int_0^1 r(x) dx \right| \leq \left| \int_0^1 r(x) \cos 2\sqrt{\lambda_n} x dx \right| + 2 \left| \int_0^1 r(x) h(x, \lambda_n) dx \right| \quad (2.17)$$

Using integration by parts

$$\int_0^1 r(x) \cos 2\sqrt{\lambda_n} x dx = \frac{r(1) \sin 2\sqrt{\lambda_n}}{2\sqrt{\lambda_n}} - \frac{1}{2\sqrt{\lambda_n}} \int_0^1 r'(x) \sin 2\sqrt{\lambda_n} x dx \quad (2.18)$$

then

$$\left| \int_0^1 r(x) \cos 2\sqrt{\lambda_n} x dx \right| \leq \frac{|r(1)|}{2\sqrt{\lambda_n}} + \frac{1}{2\sqrt{\lambda_n}} \int_0^1 |r'(x)| dx \quad (2.19)$$

Because of (2.10) and (2.11) we obtain

$$\|Y(x, \lambda_n)\| \leq 1 + 2 \int_0^x \frac{1}{\sqrt{\lambda_n}} \|Q(t)\| \|Y(t, \lambda_n)\| dt$$

By Lemma 2.3 (Gronwall's inequality ([9])) we obtain

$$\|Y(x, \lambda_n)\| \leq \exp \left\{ \int_0^x \frac{2}{\sqrt{\lambda_n}} \|Q(t)\| dt \right\} \quad (2.20)$$

Using (2.12)(1.13) and  $\sqrt{\lambda_n} > 2Q_*$ , we have, for  $1 \leq i, j \leq 2$ ,

$$|g_{ij}(x, \lambda_n)| \leq \exp \left\{ \int_0^1 \frac{2}{\sqrt{\lambda_n}} \|Q(t)\| dt \right\} - 1 \quad (2.21)$$

By the inequality  $e^x - 1 \leq xe$ , we obtain

$$|g_{ij}(x, \lambda_n)| \leq \frac{2e}{\sqrt{\lambda_n}} \int_0^1 \|Q(t)\| dt = \frac{2eQ_*}{\sqrt{\lambda_n}} \quad (2.22)$$

then

$$|h(x, \lambda_n)| \leq \frac{8e^2 Q_*}{\sqrt{\lambda_n}} \quad (2.23)$$

therefor

$$\left| \int_0^1 r(x) h(x, \lambda_n) dx \right| \leq \frac{8e^2 Q_*}{\sqrt{\lambda_n}} \int_0^1 |r(x)| dx \quad (2.24)$$

By (2.17)(2.19)(2.24) and  $\sqrt{\lambda_n} > R_*/\left|\int_0^1 r(x) dx\right|$  we obtain

$$\begin{aligned} \left| \int_0^1 r(x) dx \right| &\leq \frac{|r(1)|}{2\sqrt{\lambda_n}} + \frac{1}{2\sqrt{\lambda_n}} \int_0^1 |r'(x)| dx + 16e^2 Q_* \frac{1}{\sqrt{\lambda_n}} \int_0^1 |r(x)| dx \\ &= \frac{1}{\sqrt{\lambda_n}} R_* < \left| \int_0^1 r(x) dx \right| \end{aligned} \quad (2.25)$$

This is a contradiction. Therefore, if  $\lambda_n$  is the eigenvalue (1.1)(1.2) which satisfies the condition  $\sqrt{\lambda_n} > M_Q$ , then  $\lambda_n$  is simple.

### 3. The intersection of the spectra of two potential equations

In this section, we will apply Theorem 2.1 to study the intersection of the spectra of two potential equations of the form (1.3). Recall that for (1.3) with potential function  $q(x)$  and boundary condition (1.4),  $\sigma(q)$  is used to denote its spectrum, i.e., the set of all eigenvalues of (1.3)(1.4),  $\sigma(Q)$  is used to denote spectrum of (1.1)(1.2), i.e., the set of all eigenvalues of (1.1)(1.2).

$$\text{Let } \Lambda(x) = \begin{pmatrix} q_1(x) & 0 \\ 0 & q_2(x) \end{pmatrix}, U(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, U^*(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

and

$$Q_\theta(x) = U^*(\theta)\Lambda(x)U(\theta) = \begin{pmatrix} q_1 \cos^2 \theta + q_2 \sin^2 \theta & (q_2 - q_1) \sin \theta \cos \theta \\ (q_2 - q_1) \sin \theta \cos \theta & q_2 \cos^2 \theta + q_1 \sin^2 \theta \end{pmatrix} \quad (3.1)$$

where  $\theta$  is a constant.

#### Theorem 3.1

Suppose  $q_1(x)$  and  $q_2(x)$  are two real-value continuous functions on  $[0,1]$ , if

$$\int_0^1 q_1(x) dx \neq \int_0^1 q_2(x) dx, \quad (3.2)$$

then  $\sigma(q_1)$  and  $\sigma(q_2)$  only have finitely many elements in common.

**Proof :** If  $Q(x) = Q_\theta(x)$  in equation (1.1), then  $r(x) = (q_2 - q_1) \sin \theta \cos \theta$ , where  $\theta$  is a constant. We choose  $\theta$  so that  $\sin \theta \cos \theta \neq 0$ . First, we will prove the following equation

$$\sigma(Q_\theta) = \sigma(q_1) \cup \sigma(q_2) \quad (3.3)$$

On the one hand, suppose  $\lambda_0 \in \sigma(q_1) \cup \sigma(q_2)$ , then  $\lambda_0 \in \sigma(q_1)$  or  $\lambda_0 \in \sigma(q_2)$ . We suppose  $\lambda_0 \in \sigma(q_1)$ ,  $y_1$  is the corresponding eigenfunction, then  $y_1$  satisfies the following equation

$$y'' + (\lambda - q_1)y = 0, \quad y'(0) = y'(1) = 0 \quad (3.4)$$

Let  $Z(x) = \begin{pmatrix} y_1(x) \\ y_1(x) \tan \theta \end{pmatrix}$ , then  $Z(x)$  satisfies (1.1)(1.2) for  $\lambda = \lambda_0, Q(x) = Q_\theta(x)$ . Then  $\lambda_0 \in \sigma(Q_\theta)$ . That is

$$\sigma(q_1) \cup \sigma(q_2) \subset \sigma(Q_\theta) \quad (3.5)$$

On the other hand, if  $\lambda_0 \in \sigma(Q_\theta)$ ,  $Z(x) = \begin{pmatrix} y_1(x) \\ y_1(x) \tan \theta \end{pmatrix}$  is the corresponding eigenfunction, then  $y_1$  satisfies the equation (3.4), then  $\lambda_0 \in \sigma(q_1)$ . Or  $\lambda_0 \in \sigma(q_2)$  and  $y_1 \tan \theta$  is the corresponding eigenfunction. Then

$$\sigma(Q_\theta) \subset \sigma(q_1) \cup \sigma(q_2) \quad (3.6)$$

By (3.5)(3.6) we obtain (3.3).

By (3.2) we have  $\int_0^1 r(x) dx = \int_0^1 (q_2 - q_1) \sin \theta \cos \theta dx \neq 0$ , By theorem 2.1 we obtain two-dimension vectorial Sturm-Liouville problem (1.1)(1.2) for  $Q(x) = Q_\theta(x)$  only have finitely many eigenvalues of multiplicity two. Then  $\sigma(Q_\theta)$  have finitely many elements in common. Because the eigenvalues of (1.3)(1.4) are all simple, by (3.3), we obtain  $\sigma(q_1) \cap \sigma(q_2)$  is a finite set.

Therefor if  $\int_0^1 q_1(x) dx \neq \int_0^1 q_2(x) dx$ , then  $\sigma(q_1)$  and  $\sigma(q_2)$  only have finitely many elements in common.

Finally we apply Theorem 3.1 to study the intersection of the spectra of two string equations.

Let  $\sigma(\rho)$  be the spectrum of the following string equation with density function  $\rho(x)$ , i.e., the set of all eigenvalues of (3.7)

$$y''(x) + \mu\rho(x)y(x) = 0, \quad y'(0) = y'(1) = 0 \quad (3.7)$$

It was well known that the eigenvalues of (3.7) are all simple. Apply Theorem 3.1 we obtain the following result.

### Theorem 3.2

Suppose  $\rho_1(x)$  and  $\rho_2(x)$  are two real-value continuous functions on  $[0,1]$ , if  $\rho_1(x)$  and  $\rho_2(x)$  satisfy the following two conditions  $\int_0^1 \rho_1^{1/2}(x)dx = \int_0^1 \rho_2^{1/2}(x)dx$ ,  $\int_0^1 \rho_1^{-10/4}(x)dx \neq \int_0^1 \rho_2^{-10/4}(x)dx$ , then  $\sigma(\rho_1)$  and  $\sigma(\rho_2)$  only have finitely many elements in common.

**Proof :** Let

$$K = \int_0^1 \rho^{1/2}(t)dt, \quad z = K^{-1} \int_0^x \rho^{1/2}(t)dt,$$

$$y(x) = \rho^{-1/4}(x)u(z), \quad \lambda = K^2\mu, \quad q(z) = -\frac{5}{16}K^2\rho^{-3}(x)$$

we have

$$y'(x) = -\frac{1}{4}\rho^{-5/4}(x)u(z) + K^{-1}\rho^{1/4}(x)u'(z) \quad (3.10)$$

$$y''(x) = \frac{5}{16}\rho^{-9/4}(x)u(z) + K^{-2}\rho^{3/4}(x)u''(z) \quad (3.11)$$

Then

$$y''(x) + \mu\rho(x)y(x) = K^{-2}\rho^{3/4}(x)(u''(z) + \frac{5}{16}K^2\rho^{-3}(x)u(z) + K^2\mu u(z)) = 0$$

That is

$$u''(z) + (\lambda - q(z))u(z) = 0 \quad (3.12)$$

That is the equation (3.7) is transformed to the form of equation (1.3).

By the method of changing variables, we find that

$$\int_0^1 q(z)dz = -\frac{5}{16}K \int_0^1 \rho^{-10/4}(x)dx \quad (3.13)$$

Therefor, if  $\rho_1(x)$  and  $\rho_2(x)$  satisfy (3.8)(3.9), then

$$\int_0^1 q_1(z)dz \neq \int_0^1 q_2(z)dz,$$

By Theorem 3.1 we have  $\sigma(\rho_1)$  and  $\sigma(\rho_2)$  only have finitely many elements in common.

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