

Some Fixed Point Theorems in G-Metric and Fuzzy Metric Spaces using E.A Property

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Abstract. Fuzzy logic is a new scientific field which is employed in mathematics, computer science and Engineering. Fuzzy logic calculates the extent to which a proposition is correct and allows computer to manipulate the information. The aim of this paper is to prove some common fixed point theorems using the property E.A. in the setting of G-metric and Fuzzy metric space by taking a set of three conditions for self mappings. Here the notion of semi compatibility and occasionally compatibility is used to replace the assumption of continuity and completeness of space. Our result generalizes the result of Tanmony Som [1] and Mujahid Abbas [2].

Keywords: Fixed Point, Property E.A, G-metric space, Fuzzy metric space, compatibility

1. Introduction

The concept of Fuzzy is initially given by Lofti A Zadeh [3]. Fuzzy sets and Fuzzy logic are powerful mathematical tools for modelling and controlling uncertain systems in industry, humanity, nature and easily be implemented on a standard computer. Fuzzy metric space and fixed point in this space have been studied by many authors [4]-[7]. In 2007 R.K Saini [8] gave fixed point result in fuzzy metric space using expansion mapping. Further R-weakly commuting mapping has been employed by R.K Saini et. al. to obtain results in [9].

Mustafa [10] gave generalized metric called G-metric and define many concepts like convergence, continuity, completeness, compactness, product of spaces in the setting of G-metric space and stated that every G-metric space is topologically equivalent to a metric space. Again in 2009 Mustafa [11] showed fixed point results without using the completeness property but with other sufficient conditions. Further he employ a new concept of expansive mapping to prove results in [12].

In 1986, Jungck [13] introduced the concept of compatible mapping. This concept was frequently used to prove existence theorems in common fixed point theory. Vishal Gupta et. al. [14] gave a result in G-metric space using weakly compatibility.

In 2002, Aamir and Moutawakil [16] studied a new property for pair of maps, the property E.A, which is generalization of concept of non compatible maps. Further Pant and Pant [17] studied the common fixed points of a pair of non compatible maps and property E.A in Fuzzy metric space. Recently, Vishal Gupta and A. Kanwar[15] proved Fixed Point Theorem in Fuzzy Metric Spaces Satisfying E.A Property.

The purpose of this paper is to prove common fixed point results for four self mappings using E.A property and compatibility.

Definition 1.1: Let X be a non empty set and let $G : X \times X \times X \rightarrow R^+$ be a function satisfying the following,

- 1) $G(x, y, z) = 0$ if $x = y = z$
- 2) $0 > G(x, x, y)$ for all $x, y \in X$ with $x \neq y$
- 3) $G(x, x, y) < G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$
- 4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$
- 5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$

Then the function G is called a generalized metric or a G-metric on X and the pair (X, G) is called G-metric space.

Definition 1.2: Let A and S be self mappings on a G-metric space (X, G) . Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.3: A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if $*$ is satisfying the following conditions:

- (i) $*$ is commutative and associative
- (ii) $*$ is continuous
- (iii) $a * 1 = a$ for all $a \in [0,1]$
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 1.4: The 3-tuple $(X, M, *)$ is said to be Fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

- (i) $M(x, y, 0) = 0$
- (ii) $M(x, y, t) = 1 \quad \forall t > 0$ if and only if $x = y$
- (iii) $M(x, y, t) = M(y, x, t)$
- (iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
- (v) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous
- (vi) $\lim_{n \rightarrow \infty} M(x, y, t) = 1$

Definition 1.5: Two self mappings A and S of a Fuzzy metric space $(X, M, *)$ are said to be compatible if and only if $M(ASx_n, SAx_n, t) \rightarrow 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \rightarrow p$ for some p in X as $n \rightarrow \infty$.

Definition 1.6: Two self mappings A and S of a Fuzzy metric space $(X, M, *)$ are said to be semi-compatible if and only if $M(ASx_n, Sp, t) \rightarrow 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \rightarrow p$ for some p in X as $n \rightarrow \infty$.

Definition 1.7: Two self mappings A and S of a Fuzzy metric space $(X, M, *)$ are said to be occasionally weakly compatible if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Definition 1.8: Two self mappings A and S of a Fuzzy metric space $(X, M, *)$ are said to satisfy the property E.A if there exists sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$ for some $z \in X$.

Theorem 1.1[1]: Let S and T be two continuous self mappings of a complete fuzzy metric space $(X, M, *)$. Let A be a self mapping of X satisfying (A, S) and (A, T) are R-weakly commuting and

$$A(X) \subseteq S(X) \cap T(X) \quad \text{and} \quad (1.1)$$

$$M(Ax, Ay, t) \geq r \left[\min \{ M(Sx, Ty, t), M(Sx, Ax, t), M(Sx, Ay, t), M(Ty, Ay, t) \} \right] \quad (1.2)$$

for all $x, y \in X$, where r is a continuous function such that

$$r(t) > t \text{ for each } t < 1 \text{ and } r(t) = t \text{ for } t = 1 \quad (1.3)$$

The sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \rightarrow x, y_n \rightarrow y, t > 0$ implies $M(x_n, y_n, t) \rightarrow M(x, y, t)$. Then A, S, T have a unique common fixed point in X .

Theorem 1.2[2]: Let X be a complete G-metric space. Suppose that $\{f, S\}$ and $\{g, T\}$ be point wise R-weakly commuting pairs of self-mappings on X satisfying

$$G(fx, fx, gy) \leq h \max \{ G(Sx, Sx, Ty), G(fx, fx, Sx), G(gy, gy, Ty), [G(fx, fx, Ty) + G(gy, gy, Sx)]/2 \} \quad (1.4)$$

and

$$G(fx, gy, gy) \leq h \max \{ G(Sx, Ty, Ty), G(fx, Sx, Sx), G(gy, Ty, Ty), [G(fx, Ty, Ty) + G(gy, Sx, Sx)]/2 \} \quad (1.5)$$

for all $x, y \in X$, where $h \in (0, 1)$. Suppose that $fX \subseteq TX$, $gX \subseteq SX$ and one of the pair $\{f, S\}$ and $\{g, T\}$ is compatible. If the mappings in the compatible pair be continuous, then f, g, S and T have a unique common fixed point.