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A New Rational Quadratic Trigonometric Bézier Curve with Three Shape Parameters

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Abstract. A rational quadratic trigonometric Bézier curve with three shape parameters is presented in this work. The properties of the curve are studied. The C^0, C^1 and C^2 continuous conditions for joining two constructed curves are discussed. The shape of the curve can be flexibly controlled with shape parameters and weight without changing the control points. Some examples are given.

Keywords: quadratic trigonometric basis functions, rational quadratic trigonometric Bézier curve, shape parameter, continuity.

1. Introduction

Trigonometric splines have gained widespread application in various fields of mathematics, physics and engineering, in particular in curve design (cf. [1-6]). In recent years, Bézier form of trigonometric curvers with shape parameters has received very much attention in Computer Aided Geometric Design (CAGD). For example, Han [7-8] proposed quadratic trigonometric Bézier curves and cubic trigonometric Bézier curves with a shape parameter. Han et al [9] presented the cubic trigonometric Bézier curve with two shape parameters. Sheng et al [10] introduced the quasi-quartic Bézier-type curves with parameter α . Bashir et al [11] gave a class of quasi-quintic trigonometric Bézier curve with two shape parameters.

In this paper, we define a new rational quadratic trigonometric Bézier curve with three shape parameters. It is more flexible to control the shape than the presented curve in [12]. The composition of two curve segments using C^0, C^1 and C^2 continuity conditions is discussed. Some examples illustrate that the constructed curve in this paper provides an effective method for designing curves and geometric modeling.

2. Basis Functions

Definition 1. For $t \in [0,1]$, the quadratic trigonometric basis functions with three shape parameters α, β and γ are defined as

$$\begin{cases} b_0(t) = (1-\sin\frac{\pi}{2}t)(1-\alpha\sin\frac{\pi}{2}t)(1-\beta\sin\frac{\pi}{2}t) \\ b_1(t) = 1 - b_0(t) - b_2(t) \\ b_2(t) = (1-\cos\frac{\pi}{2}t)(1-\alpha\cos\frac{\pi}{2}t)(1-\gamma\cos\frac{\pi}{2}t) \end{cases} \quad (1)$$

where $\alpha, \beta, \gamma \in [-1,1]$ and satisfy that α with β and γ cannot be simultaneously negative.

Theorem 1. The basis functions have the following properties:

(i) Non-negativity: $b_i(t) \ge 0$, $i = 0,1,2$.

(ii) Partitin of unity: $\sum_{i=0}^{2} b_i(t) \equiv 1$.

(iii) Symmetry: $b_i(t;\alpha,\beta,\gamma) = b_i(1-t;\alpha,\gamma,\beta)$, $i = 0,1,2$.

(iv) Monotonicity: For fixed $t \in [0,1]$, $b_0(t)$ is monotonically decreasing for shape parameters α and β. $b_2(t)$ is monotonically decreasing for shape parameters α and γ. $b_1(t)$ is monotonically increasing respectively for shape parameters α, β and γ.

(v) Properties at the endpoints:

$$b_0(0)=1,\quad b_1(0)=0,\quad b_2(0)=0;$$
$$b_0(1)=0,\quad b_1(1)=0,\quad b_2(1)=1;$$
$$b_0^{'}(0)=-\frac{\pi}{2}(1+\alpha+\beta),\quad b_1^{'}(0)=\frac{\pi}{2}(1+\alpha+\beta),\quad b_2^{'}(0)=0;$$
$$b_0^{'}(1)=0,\quad b_1^{'}(1)=-\frac{\pi}{2}(1+\alpha+\gamma),\quad b_2^{'}(0)=\frac{\pi}{2}(1+\alpha+\gamma).$$

Proof. The results immediately follow from the defintion of the basis functions (1).

Fig. 1 shows the curves of the quadratic trigonometric basis functions for $\alpha=0,\beta=-1,\gamma=1$ (solid lines), for $\alpha=1,\beta=0,\gamma=0$ (short dashed lines), and $\alpha=-1,\beta=1/2,\gamma=1$ (long dashed lines), respectively.

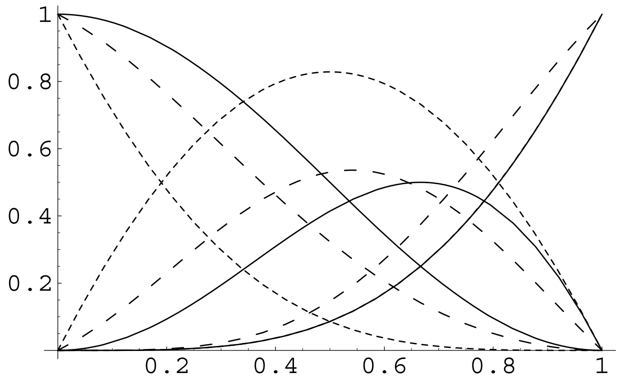

Fig. 1: The quadratic trigonometric basis functions.

3. The RQTB Curve

Definition 2. Given that $P_i(i=0,1,2)$ are three control points in $R^d(d=2,3)$,

$$R(t)=\frac{b_0(t)P_0+b_1(t)P_1\omega+b_2(t)P_2}{b_0(t)+b_1(t)\omega+b_2(t)},\quad t\in[0,1] \tag{2}$$

is called the rational quadratic trigonometric Bézier (RQTB, for short) curve with three shape parameters α, β and γ, where $\omega(>0)$ is called the weight of the function, and the basis functions $b_i(t)$ are defined as (1).

Theorem 2. The RQTB curve has the following properties:

(i) Terminal properties:

$$
\begin{aligned}
&R(0)=P_0, \quad R(1)=P_2,\\
&R'(0)=\frac{\pi}{2}(1+\alpha+\beta)\omega(P_1-P_0), \quad R'(1)=\frac{\pi}{2}(1+\alpha+\gamma)\omega(P_2-P_1),\\
&R''(0)=\frac{\pi^2}{4}((1-\alpha)(1-\gamma)(P_2-P_0)\\
&-(2(1+\alpha+\beta)^2-2(\alpha+\beta)^2-2\alpha\beta-(1-\alpha)(1-\gamma))\omega(P_0-P_1)\\
&+2(1+\alpha+\beta)^2\omega^2(P_0-P_1)),\\
&R''(1)=\frac{\pi^2}{4}((1-\alpha)(1-\beta)(P_0-P_2)\\
&-(2(1+\alpha+\gamma)^2-2(\alpha+\gamma)^2-2\alpha\gamma-(1-\alpha)(1-\beta))\omega(P_2-P_1)\\
&+2(1+\alpha+\gamma)^2\omega^2(P_2-P_1)).
\end{aligned}
\tag{3}
$$

(ii) Symmetry: If the weight ω is kept fixed, P_0, P_1, P_2 and P_2, P_1, P_0 define the same RQTB curve in different parameterizations, i.e.,

$$R(t;\alpha,\beta,\gamma,P_0,P_1,P_2)=R(1-t;\alpha,\gamma,\beta,P_2,P_1,P_0),$$

where $t\in[0,1]$, $\alpha,\beta,\gamma\in[-1,1]$, α with β and γ cannot be simultaneously negative.

(iii) Geometric invariance: The shape of the RQTB curve is independent of the choice of coordinates, i.e.,

$$
\begin{aligned}
&R(t;P_0+q,P_1+q,P_2+q)=R(t;P_0,P_1,P_2)+q,\\
&R(t;P_0*T,P_1*T,P_2*T)=R(t;P_0,P_1,P_2)*T,
\end{aligned}
$$

where q is an arbitrary vector in R^2 or R^3, and T is an arbitrary $m\times m$ matrix, $m=2$ or 3.

(iv) Convex hull property: The RQTB curve segment must lie inside its control polygon spanned by P_0, P_1, P_2.

Remark 1. If $\omega=1$, the curve (2) will be called a quadratic trigonometric Bézier curve with three shape parameters. If $\alpha=0$, the RQTB curve will reduce to the rational quadratic trigonometric Bézier curve with two shape parameters in [12]. For $\alpha=-1,\beta=\gamma=0$ or $\alpha=0,\beta=\gamma=-1$, the RQTB curve will reduce to $R(t)=\cos^2(\frac{\pi}{2}t)P_0+\sin^2(\frac{\pi}{2}t)P_2, t\in[0,1]$, which is a straight line between the control points P_0 and P_2.

4. Continuity of the Curves

Let a RQTB curve $R(t)$ be given as (2) and a second RQTB curve $R^*(t)$ be defined by

$$R^*(t)=\frac{b_0(t)P_0^*+b_1(t)P_1^*\omega^*+b_2(t)P_2^*}{b_0(t)+b_1(t)\omega^*+b_2(t)}, \quad t\in[0,1] \tag{4}$$

where the weight $\omega^*>0$, three shape parameters $\alpha^*,\beta^*,\gamma^*\in[-1,1]$ and satisfy that α^* with β^* and γ^* cannot be simultaneously negative.

Theorem 3. Given Two segments of RQTB curves $R(t)$ and $R^*(t)$, then the necessary and sufficient condition of continuity is

(i) for C^0 continuity,

$$P_0^*=P_2;$$

(ii) for C^1 continuity,

$$P_1^* - P_0^* = \frac{(1+\alpha+\gamma)\omega}{(1+\alpha^*+\beta^*)\omega^*}(P_2 - P_1);$$

(iii) for C^2 continuity,

$$P_2^* = P_0 + 2\omega(\omega+\omega^*)(P_2 - P_1)$$

with $\alpha = \beta = \gamma = \alpha^* = \beta^* = \gamma^* = 0$.

Proof. The result (i) is obvious for $R^*(0) = R(1)$.

For C^1 continuity, the tangents of the two curves at the joint must be equal, that is,

$$\begin{cases} P_0^* = P_2, \\ R^{*}{'}(0) = R'(1). \end{cases}$$

Then

$$\frac{\pi}{2}(1+\alpha^*+\beta^*)\omega^*(P_1^* - P_0^*) = \frac{\pi}{2}(1+\alpha+\gamma)\omega(P_2 - P_1).$$

The result (ii) holds after simple reorganization.

The two curvers are joined by C^2 continuity if

$$\begin{cases} P_0^* = P_2, \\ R^{*}{'}(0) = R'(1), \\ R^{*}{''}(0) = R''(1). \end{cases}$$

Taking $\alpha = \beta = \gamma = \alpha^* = \beta^* = \gamma^* = 0$, we obtain from (3)

$$P_2^* - P_0^* + 2(\omega^*)^2(P_0^* - P_1^*) = P_0 - P_2 + 2\omega^2(P_2 - P_1).$$

Since $\omega^*(P_1^* - P_0^*) = \omega(P_2 - P_1)$ and $P_0^* = P_2$, the result (iii) follows.

5. Shape Control of the RQTB Curve

Given control points, the shape of RQTB curve can be adjusted by alerting the values of the shape parameters α, β and γ. Moreover the weight ω offers an additional control on the curve.

In Fig.2(a), the curves are generated by setting $\beta = \gamma = 0, \omega = 2$ and changing α to $\alpha = -1/2$ (long dashed lines) and $\alpha = 0$ (solid lines) and $\alpha = 1$ (short dashed lines), respectively.

In Fig.2(b), the curves are generated by setting $\alpha = \gamma = 0, \omega = 2$ and changing β to $\beta = -1/2$ (long dashed lines) and $\beta = 0$ (solid lines) and $\beta = 1$ (short dashed lines), respectively.

In Fig.2(c), the curves are generated by setting $\alpha = \beta = 0, \omega = 2$ and changing γ to $\gamma = -1/2$ (long dashed lines) and $\gamma = 0$ (solid lines) and $\gamma = 1$ (short dashed lines), respectively.

In Fig.2(d), the curves are generated by setting $\alpha = \beta = \gamma = 0$ and changing ω to $\omega = 1$ (long dashed lines) and $\omega = 2$ (solid lines) and $\omega = 3$ (short dashed lines), respectively.

The presence of shape parameters and the weight provides an intuitive control on the shape of the curve. Fig.3 and Fig.4 show two graphs of pattern. The RQTB curves are generated by setting $\alpha = \beta = \gamma = 0$, $\omega = 2$ (solid lines), $\alpha = \beta = \gamma = 0$, $\omega = 1$ (short dashed lines), $\alpha = 3/4$, $\beta = \gamma = 0$, $\omega = 1$(long dashed lines).

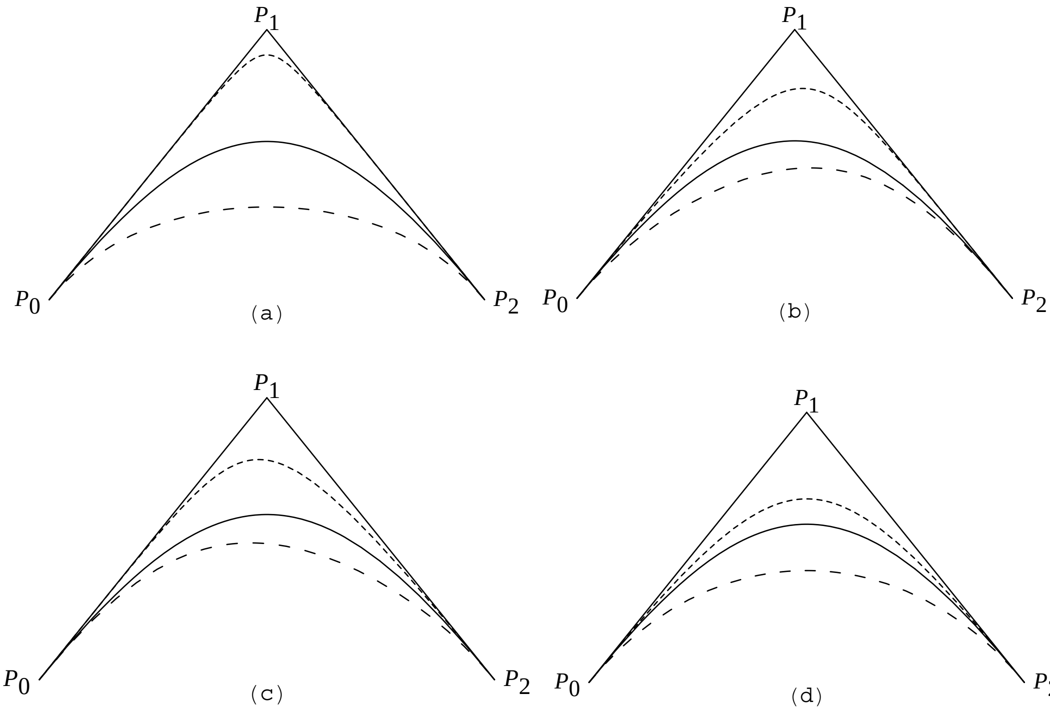

Fig. 2: The effects on the shape of RQTB curve for different values of α, β, γ and ω.

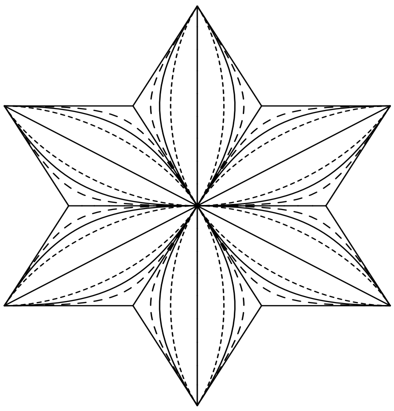

Fig. 3: Ornamental pattern

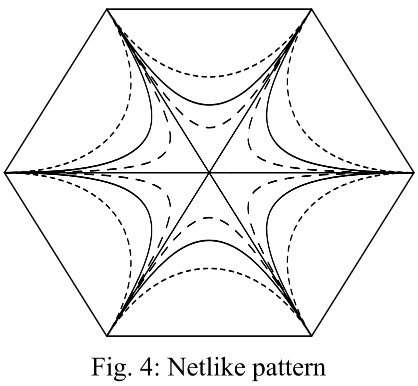

Fig. 4: Netlike pattern

6. Conclusion

The rational quadratic trigonometric Bézier curve with three shape parameters is presented. The proposed curve enjoys all the geometric properties of the traditional rational quadratic Bézier curve. The shape of the curve can be adjusted by altering the values of shape parameters while the control polygon is kept unchanged. Moreover the weight offers an additional control on the curve. It can be freely adopted in CAD/CAM systems..

7. References

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